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On Open and Closed Convex Codes

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Abstract

Neural codes serve as a language for neurons in the brain. *Open* (*or closed*) *convex codes*, which arise from the pattern of intersections of collections of open (or closed) convex sets in Euclidean space, are of particular relevance to neuroscience. Not every code is open or closed convex, however, and the combinatorial properties of a code that determine its realization by such sets are still poorly understood. Here we find that a code that can be realized by a collection of open convex sets may or may not be realizable by closed convex sets, and vice versa, establishing that *open convex* and *closed convex* codes are distinct classes. We establish a non-degeneracy condition that guarantees that the corresponding code is both open convex and closed convex. We also prove that *max intersection-complete* codes (i.e., codes that contain all intersections of maximal codewords) are both open convex and closed convex, and provide an upper bound for their minimal embedding dimension. Finally, we show that the addition of non-maximal codewords to an open convex code preserves convexity.

Keywords Convex codes · Intersection-complete codes · Neural codes · Combinatorial codes · Embedding dimension

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1 Introduction

The brain represents information via patterns of neural activity. Often, one can think of these patterns as strings of binary responses, where each neuron is "on" or "off" according to whether or not a given stimulus lies inside its receptive field. In this scenario, the *receptive field* $U_i \subset X$ of a neuron i is simply the subset of stimuli to which it responds, with X being the entire stimulus space. A collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of receptive fields for a population of neurons $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$ gives rise to the combinatorial code[1]

$$\operatorname{code}(\mathcal{U}, X) \stackrel{\text{def}}{=} \left\{\sigma \subseteq [n] \text{ such that } A_\sigma^{\mathcal{U}} \neq \varnothing\right\} \subseteq 2^{[n]},$$

where $2^{[n]}$ is the set of all subsets of $[n]$, and the *atoms* $A_\sigma^{\mathcal{U}}$ correspond to regions of the stimulus space carved out by $\mathcal{U}$:

$$A_\sigma^{\mathcal{U}} \stackrel{\text{def}}{=} \left(\bigcap_{i\in\sigma} U_i\right) \setminus \bigcup_{j\notin\sigma} U_j \subseteq X.$$

Here every stimulus $x \in A_\sigma^{\mathcal{U}}$ gives rise to the same neural response pattern, or codeword, $\sigma \subseteq [n]$. By convention, $\bigcap_{i\in\varnothing} U_i = X$ and thus $A_\varnothing^{\mathcal{U}} = X \setminus \left(\bigcup_{i=1}^n U_i\right)$, so that $\varnothing \in \operatorname{code}(\mathcal{U}, X)$ if and only if $\bigcup_{i=1}^n U_i \neq X$. Note that $\operatorname{code}(\mathcal{U}, X)$ may fail to be an abstract simplicial complex; see e.g., Fig. 1.

Definition 1.1 We say that a combinatorial code $\mathcal{C} \subseteq 2^{[n]}$ is *open convex* if $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$ for a collection $\mathcal{U} = \{U_i\}_{i=1}^n$ of open convex subsets $U_i \subseteq X \subseteq \mathbb{R}^d$ for some $d \geq 1$. Similarly, we say that $\mathcal{C}$ is *closed convex* if $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$ for a collection of closed convex subsets $U_i \subseteq X \subseteq \mathbb{R}^d$. For an open convex code $\mathcal{C}$, the *embedding dimension* $\operatorname{odim}\mathcal{C}$ is the minimal d for which there exists an open convex realization of $\mathcal{C}$ as $\operatorname{code}(\mathcal{U}, X)$. Similarly, for a closed convex code $\operatorname{cdim}\mathcal{C}$ is the minimal d that admits a closed convex realization of $\mathcal{C}$.

Note that if the condition that all sets are open, or alternatively all sets are closed, is relaxed, then there are no known obstructions for a code to arise as a code of a convex cover. In particular, it has been proved in [5] that any code can be realized as a code of a cover by arbitrary convex sets. For this reason, we only consider either open or closed convex codes.

Open and closed convex codes have special relevance to neuroscience because neurons in a number of areas of mammalian brains possess convex receptive fields. A paradigmatic example is that of hippocampal *place cells* [12], a class of neurons in the hippocampus that act as position sensors. Here the relevant stimulus space $X \subset \mathbb{R}^d$ is the animal's environment [14], with $d \in \{1, 2, 3\}$. Receptive fields can be easily

[1] A *combinatorial code* is any collection of subsets $\mathcal{C} \subseteq 2^{[n]}$. Each $\sigma \in \mathcal{C}$ is called a *codeword*.

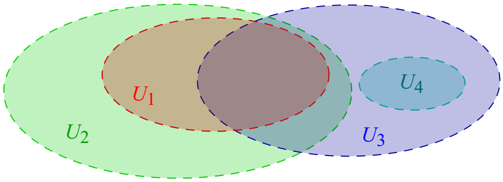


Fig. 1 An example of a cover $\mathcal{U} = \{U_i\}$ and its code, $\mathcal{C} = \text{code}(\mathcal{U}, X) = \{\varnothing, 2, 3, 12, 23, 34, 123\}$, where $X = \mathbb{R}^2$. Here we denote a codeword $\{i_1, i_2, \ldots, i_k\} \in \mathcal{C}$ by the string $i_1 i_2 \ldots i_k$; for example, $\{1, 2, 3\}$ is abbreviated to 123. Since $13 \notin \mathcal{C}$ but $13 \subset 123$, $\mathcal{C}$ is not a simplicial complex

computed when both the neuronal activity data and the relevant stimulus space are available. However, in many situations the relevant stimulus space for a given neural population may be unknown. This raises the natural question: how can one determine from the intrinsic properties of a combinatorial code whether or not it is an open (or closed) convex code? What is the embedding dimension of a code – that is, what is the dimension of the relevant stimulus space? How are open and closed convex codes related?

The code of a cover carries more information about the geometry/topology of the underlying space than the nerve of the cover. For example, it imposes more constraints on the embedding dimension than what is imposed by the nerve [3]. Arrangements of convex sets are ubiquitous in applied and computational topology, however all the standard constructions (e.g., the Čech complex) rely only on the nerve of the cover, and do not carry any information about the arrangement beyond the nerve. While the properties of nerves of convex covers were previously studied in [8,9,13], codes of convex covers are much less understood. Moreover, although any simplicial complex can be realized as the nerve of an open or closed convex cover (in high enough dimension), not all combinatorial codes can be realized from arrangements of open or closed convex sets in Euclidean space.

There is currently little understanding of what makes a code open or closed convex beyond 'local obstructions' to convexity [2,6]. Furthermore, local obstructions can only be used to show that a code is *not* open or closed convex, and the absence of local obstructions does not guarantee open or closed convexity of the code [11]. To show that a code *is* open or closed convex, one must produce a realization, and there are few results that guarantee such an open or closed convex realization exists. Our first main result makes significant progress in this regard, as it provides a general condition for determining that a code is convex from combinatorial properties alone. Specifically, we show that *max intersection-complete* codes—i.e., codes that contain all intersections of their maximal[2] codewords—are both open convex and closed convex.

Theorem 1.2 *Suppose $\mathcal{C} \subset 2^{[n]}$ is a max intersection-complete code. Then $\mathcal{C}$ is both open convex and closed convex. Moreover, the embedding dimensions satisfy* $\operatorname{odim} \mathcal{C} \leq \max\{2, (k-1)\}$ *and* $\operatorname{cdim} \mathcal{C} \leq \max\{2, (k-1)\}$, *where k is the number of maximal codewords of $\mathcal{C}$.*

The fact that max intersection-complete codes are open convex was first hypothesized in [2], where it was shown that these codes have no local obstructions. In our

[2] A codeword in $\sigma \in \mathcal{C}$ is *maximal* if, as a subset $\sigma \subseteq [n]$, it is not contained in any other codeword of $\mathcal{C}$.

proof we provide an explicit construction of the convex realizations and the upper bound for the corresponding embedding dimensions. This dimension bound is tight. It is achieved on the "deleted simplex code" $\mathcal{C} = 2^{[k]} \setminus [k]$, since Helly's theorem forbids a realization of this code by convex sets in $\mathbb{R}^d$ for any $d < k - 1$.

Our next main result shows that open convex codes exhibit a certain type of *monotonicity*, in the sense that adding non-maximal codewords to an open convex code preserves convexity.

Theorem 1.3 *Assume that a code $\mathcal{C} \subset 2^{[n]}$ is open convex. If $\mathcal{D} \supset \mathcal{C}$ has the same maximal codewords as $\mathcal{C}$, then $\mathcal{D}$ is also open convex and has embedding dimension* $\operatorname{odim} \mathcal{D} \leq \operatorname{odim} \mathcal{C} + 1$.

It is currently unknown if the monotonicity property holds for the closed convex codes.

It has been recently shown in [5] that any code can be realized by a convex cover in the absence of any other constraints imposed on the convex sets in the cover. This highlights the importance of the "open" or "closed" conditions in Definition 1.1 and raises a natural question: what is the relationship of the open and the closed convex codes?

Here we establish that open convex codes and closed convex codes are indeed distinct classes of codes, that is, one class is not a sub-class of the other. This result suggests that combinatorial properties of convex codes are richer than previously believed. This finding motivates us to find a non-degeneracy condition on the cover that guarantees that the corresponding code is both open convex and closed convex. Theorem 2.12 in Sect. 2.3 establishes that this condition is sufficient. It is also used as one of the main ingredients in the proof of Theorem 1.2. We propose that codes that are both open convex and closed convex are the most relevant to neuroscience, as the intrinsic noise in neural responses makes the difference between open and closed receptive fields unobservable [10].

2 Convex Codes

We begin with observing that without sufficiently strong assumptions about the cover $\mathcal{U} = \{U_i\}$, any code can be realized as $\operatorname{code}(\mathcal{U}, X)$.

Lemma 2.1 *Every code $\mathcal{C} \subset 2^{[n]}$ can be obtained as $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$ for a collection of (not necessarily convex) $U_i \subset \mathbb{R}^1$.*

Proof It suffices to consider the case where each $i \in [n]$ appears in some codeword $\sigma \in \mathcal{C}$. For each $\sigma \in \mathcal{C}$, choose points $x_\sigma \in \mathbb{R}^1$ such that $x_\sigma \neq x_\tau$ if $\sigma \neq \tau$. Define $U_i = \{x_\sigma \mid i \in \sigma\}$ and $\mathcal{U} = \{U_i\}_{i \in [n]}$. If $\varnothing \in \mathcal{C}$, then $\mathcal{C} = \operatorname{code}(\mathcal{U}, \mathbb{R}^1)$. Otherwise, $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$, where $X = \bigcup_{\sigma \in \mathcal{C}} \{x_\sigma\}$. □

The sets U_i in the above proof are finite subsets of $\mathbb{R}^1$. However, even if one requires that the sets U_i be open and connected, almost all codes can still arise as the code of such cover.

Lemma 2.2 *Any code $\mathcal{C} \subset 2^{[n]}$ that contains all singleton codewords, i.e., $\forall i \in [n]$, $\{i\} \in \mathcal{C}$, can be obtained as $\mathcal{C} = \text{code}(\mathcal{U}, X)$ for a collection of open connected subsets $U_i \subset \mathbb{R}^3$.*

Proof Similar to the proof of Lemma 2.1, one can place disjoint open balls $B_\sigma \subset \mathbb{R}^3$ for each $\sigma \in \mathcal{C}$ and define $U_i = \left(\bigcup_{i\in\sigma} B_\sigma\right) \cup T_i$, where each $T_i \subset \mathbb{R}^3$ is a collection of open "narrow tubes" that connect all the balls B_σ with $\sigma \ni i$. Because these sets are embedded in $\mathbb{R}^3$, the "tubes" T_i can always be arranged so that for each $i \neq j$ the intersections $T_i \cap T_j$ are contained in the union of the balls B_σ. Take $X = \left(\bigcup_{i=1}^n U_i\right) \cup B_\varnothing$ if $\varnothing \in \mathcal{C}$, or $X = \bigcup_{i=1}^n U_i$ if not. By construction, the U_i are connected and open, and $\mathcal{C} = \text{code}(\mathcal{U}, X)$. □

The condition of having all singleton words can *not* be relaxed without any further assumptions. For example, it can be easily shown that the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$, previously described in [3,6] cannot be realized as a code of a cover by open connected sets.[3]

2.1 Known Local Obstructions to Convexity

Any combinatorial code $\mathcal{C} \subset 2^{[n]}$ can be completed to an abstract simplicial complex $\Delta(\mathcal{C})$, the *simplicial complex of the code*, which is the minimal simplicial complex containing $\mathcal{C}$.[4] Note that $\Delta(\mathcal{C})$ is determined solely by the maximal codewords of $\mathcal{C}$ (facets of $\Delta(\mathcal{C})$). A code can thus be thought of as a simplicial complex with some of its non-maximal faces "missing". Moreover, given a collection of sets $\mathcal{U}$ and X, one can easily see that the simplicial complex of $\text{code}(\mathcal{U}, X)$ is equal to the usual *nerve* of the cover $\mathcal{U}$:

$$\Delta(\text{code}(\mathcal{U}, X)) = \text{nerve}(\mathcal{U}) \overset{\text{def}}{=} \left\{\sigma \subseteq [n] \text{ such that } \bigcap_{i\in\sigma} U_i \neq \varnothing\right\}.$$

For example, Fig. 1 depicts a code of the form $\mathcal{C} = \text{code}(\mathcal{U}, X)$ that differs from its simplicial complex $\Delta(\mathcal{C})$ because the subset $\{1, 3\}$ is missing. This results from the fact that $U_1 \cap U_3 \subseteq U_2$, a set containment that is not encoded in $\text{nerve}(\mathcal{U})$.

Not every code arises from a closed convex or open convex cover. For example, the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$ above cannot be an open (or closed) convex code. The failure of this code to be open or closed convex is "local" in that it is missing the codeword 3, and adding new codewords which do not include $i = 3$ would not make this code open or closed convex.

In the setting of simplicial complexes, such locality is often characterized by studying an auxiliary simplicial complex called the *link* of a simplex. This notion can be directly generalized for codes.

[3] Indeed, assuming the converse, it follows that $U_3 = (U_1 \cap U_3) \cup (U_2 \cap U_3)$ and, since this code does not contain a codeword $\sigma \supseteq 12$, we conclude that $U_1 \cap U_2 = \varnothing$ and U_3 is a union of two disjoint open sets, which yields a contradiction.

[4] Throughout, we assume that $\varnothing$ is an element of every abstract simplicial complex.

Definition 2.3 For any $\sigma \subset [n]$ the *link* of $\mathcal{C}$ at σ is the code $\operatorname{link}_\sigma \mathcal{C} \subseteq 2^{[n]\setminus\sigma} \subset 2^{[n]}$ on the same set of neurons, defined as

$$\operatorname{link}_\sigma \mathcal{C} \stackrel{\text{def}}{=} \left\{\tau \in 2^{[n]\setminus\sigma} \,\middle|\, \tau \cup \sigma \in \mathcal{C} \text{ and } \tau \cap \sigma = \varnothing\right\}.$$

For codes, the link is typically *not* a simplicial complex, but because both $\operatorname{link}_\sigma \mathcal{C}$ and $\operatorname{link}_\sigma \Delta(\mathcal{C})$ have the same set of maximal elements,

$$\Delta(\operatorname{link}_\sigma \mathcal{C}) = \operatorname{link}_\sigma \Delta(\mathcal{C}).$$

Moreover, it is easy to see that if $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$, then for every non-empty $\sigma \in \Delta(\mathcal{C})$

$$\operatorname{link}_\sigma \mathcal{C} = \operatorname{code}\big(\{U_j \cap U_\sigma\}_{j\in[n]\setminus\sigma}, U_\sigma\big), \quad \text{where} \quad U_\sigma = \bigcap_{i\in\sigma} U_i.$$

Since any intersection of convex sets is convex, we thus observe

Lemma 2.4 *If $\mathcal{C}$ is an open (or closed) convex code, then for any $\sigma \in \Delta(\mathcal{C})$, $\operatorname{link}_\sigma \mathcal{C}$ is also an open (or closed) convex code.*

Lemma 2.4 provides a framework for studying how local features of a code can obstruct realization by open or closed convex sets. Consider those faces $\sigma \in \Delta(\mathcal{C})\setminus\mathcal{C}$, which we refer to as the *simplicial violators* of $\mathcal{C}$. Suppose $\mathcal{C}$ is an open or closed convex code realized as $\operatorname{code}(\mathcal{U}, X)$, σ is a simplicial violator, and write $V_j = U_j \cap U_\sigma$. Then the code $\operatorname{link}_\sigma \mathcal{C} = \operatorname{code}(\{V_i\}, U_\sigma)$ is special in that the convex sets V_j cover another convex set U_σ. This appearance of a good cover suggests a topological approach, via a special case of the *nerve lemma*.

Lemma 2.5 (Nerve Lemma, [1,4]) *For any finite cover $\mathcal{V} = \{V_i\}_{i\in[n]}$ by convex sets $V_i \subset \mathbb{R}^d$ that are either all open or all closed,*[5] *the abstract simplicial complex*

$$\operatorname{nerve}(\mathcal{V}) \stackrel{\text{def}}{=} \left\{\sigma \subseteq [n] \textit{ such that } \bigcap_{i\in\sigma} V_i \neq \varnothing\right\} \subset 2^{[n]},$$

known as the nerve *of the cover is homotopy equivalent to the underlying space $X = \bigcup_{i\in[n]} V_i$.*

Since all convex sets are contractible, a simple corollary of Lemma 2.4 and the nerve lemma is the following observation (which first appeared in [6]), that provides a class of *local obstructions* to a code being open (or closed) convex.

Proposition 2.6 *Let $\sigma \neq \varnothing$ be a simplicial violator of a code $\mathcal{C}$. If $\operatorname{link}_\sigma \Delta(\mathcal{C})$ is not a contractible simplicial complex, then $\mathcal{C}$ is not an open (or closed) convex code.*

[5] A formulation of the nerve lemma which applies to finite collections of closed, convex subsets of Euclidean space appears in [4], and follows from [1, Thm. 10.7].

Proof Assume the converse, i.e., $\mathcal{C}$ is open (or closed) convex and σ a simplicial violator of $\mathcal{C}$. Then the sets $U_j \cap U_\sigma$ cover a convex and open (or closed) set U_σ, and thus by the nerve lemma the simplicial complex

$$\text{nerve}\big(\{U_j \cap U_\sigma\}_{j\in[n]\setminus\sigma}\big) = \Delta(\text{link}_\sigma\,\mathcal{C}) = \text{link}_\sigma\,\Delta(\mathcal{C})$$

is contractible. □

As an example, consider $\mathcal{C} = \{\varnothing, 1, 2, 3, 4, 123, 124\}$. Then $\sigma = 12$ is a simplicial violator of $\mathcal{C}$ and $\text{link}_\sigma\,\mathcal{C} = \{3, 4\}$. Since $\Delta(\text{link}_\sigma\,\mathcal{C})$ is not contractible, the code $\mathcal{C}$ is not the code of an open (or closed) convex cover. This is perhaps the minimal example of a code that cannot be realized by open or closed convex sets but can be realized by an open cover by connected sets. In fact, all the codes which are neither open nor closed convex on three neurons (these were classified in [3]) cannot be realized by open (or closed) connected sets. This is because the only obstruction to open or closed convexity is the "disconnection" of one set, similar to the case of the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$.

2.2 Do Truly "Non-local" Obstructions Via Nerve Lemma Exist?

The local obstructions to open or closed convexity in Proposition 2.6 equally apply to any open or closed good cover, i.e., a cover where each non-empty intersection $U_\sigma = \bigcap_{i\in\sigma} U_i$ is contractible. Since this property stems from applying the nerve lemma to the cover of U_σ by the other contractible sets, it is natural to define a more general *non-local* obstruction to convexity that also stems from the nerve lemma.

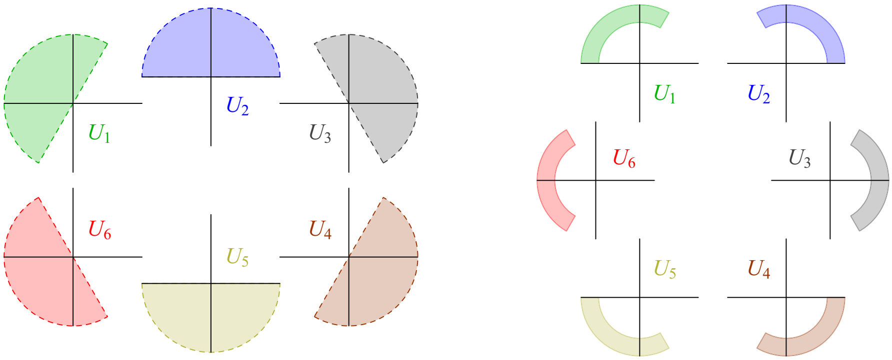
Definition 2.7 We say that a subset $\sigma \subseteq [n]$ *covers* a code $\mathcal{C} \subseteq 2^{[n]}$ if for every $\tau \in \mathcal{C}$, $\tau \cap \sigma \neq \varnothing$.

Note that any code covered by at least one set σ does not contain the empty set. Moreover, σ covers $\mathcal{C} = \text{code}\big(\{U_i\}_{i\in[n]}, \bigcup_{i\in[n]} U_i\big)$ if and only if $\bigcup_{i\in[n]} U_i = \bigcup_{j\in\sigma} U_j$; that is, $\{U_j\}_{j\in\sigma}$ is a cover of $\bigcup_{i\in[n]} U_i$.

Lemma 2.8 *If there exist two non-empty subsets $\sigma_1, \sigma_2 \subseteq [n]$ that both cover the code $\mathcal{C} \subseteq 2^{[n]}$, but the codes $\mathcal{C}\cap\sigma_a \overset{\text{def}}{=} \{\tau\cap\sigma_a \mid \tau \in \mathcal{C}\} \subseteq 2^{\sigma_a}$ for $a \in \{1, 2\}$ have simplicial complexes $\Delta(\mathcal{C} \cap \sigma_a)$ that are not homotopy equivalent, then $\mathcal{C}$ is not a code of a convex cover by open (or closed) sets in $\mathbb{R}^d$.*

This type of obstruction to open or closed convexity can be thought as non-local, as it is conditioned on the homotopy type of a complex which covers the entire code. The proof is a simple extension of the proof of Proposition 2.6, without foreknowledge of the homotopy type of the covered space.

Proof Suppose $\mathcal{C} = \text{code}\big(\mathcal{U}, \bigcup_{i\in[n]} U_i\big)$ for some open (or closed) convex sets $\mathcal{U}$. The condition that each of the non-empty subsets σ_a covers the code $\mathcal{C}$ implies that that $\bigcup_{i\in[n]} U_i = \bigcup_{j\in\sigma_a} U_j$ for each $a \in \{1, 2\}$. Thus, by the nerve lemma, $\Delta(\mathcal{C})$ has the same homotopy type as each of the complexes $\Delta(\mathcal{C} \cap \sigma_a)$, and in particular they are homotopy equivalent to one another. This yields a contradiction. □



(a) An open convex realization of (1)

(b) A closed good cover realization of (1)

Fig. 2 Two different realizations of the code $\mathcal{C}$ in (1). In both cases, the ambient space is $X = \mathbb{R}^2$. (a) The open convex realization consists of six open half-disks (or half-spaces), with opposite pairs sharing their linear boundary components. Three-element codewords correspond to the six open "pie slices", while two-element words lie in the boundaries between these. (b) The closed good realization consists of closed annular sections. Two-element codewords correspond to the interior of pairwise intersections, while three-element codewords arise from the 1-dimensional triple intersections

While it is straightforward to produce combinatorial codes with such non-local obstructions, every such code that we have considered[6] also possesses a local obstruction for open or closed convexity. Perhaps the smallest such example is the code $\mathcal{C} = \{23, 14, 123\}$. This code meets the conditions of Lemma 2.8 with $\sigma_1 = \{12\}$, and $\sigma_2 = \{34\}$, but also has a local obstruction for the simplicial violator $\sigma = \{1\}$. This provides some evidence for the conjecture that any code $\mathcal{C} \subset 2^{[n]}$ that has a non-local obstruction to open or closed convexity of the type described in Lemma 2.8 must also have a local obstruction.

2.3 The Difference Between Open and Closed Convex Codes

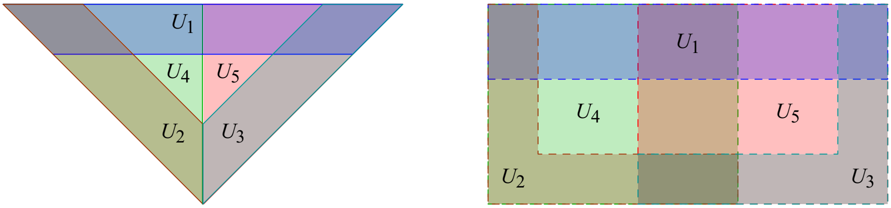
The homotopy type obstructions via the nerve lemma are obstructions to being a code of a good cover (as opposed to convex sets) and equally apply to both open and closed versions of the Definition 1.1. However, it turns out that the open and the closed convex codes are distinct classes of codes. Perhaps a minimal example of an open convex code that is not closed convex is the code

$$\mathcal{C} = \{123, 126, 156, 456, 345, 234, 12, 16, 56, 45, 34, 23, \varnothing\} \subseteq 2^{[6]}. \tag{1}$$

This code is realizable by an open convex cover (Fig. 2a) and also by an open or closed good cover (Fig. 2b).

Lemma 2.9 *The code* (1) *is not closed convex.*

[6] This included computer-assisted search among random codes.



(a) A closed convex realization of (2)

(b) An open good cover realization of (2)

Fig. 3 Two different realizations of the code in (2). In both examples, the ambient space is the union of the sets, and the four-element codeword is realized in the bottom-center in both cases

The proof is given in the Appendix (Sect. A.1). A different example,

$$\mathcal{C} = \{2345, 124, 135, 145, 14, 15, 24, 35, 45, 4, 5\} \subseteq 2^{[5]}, \tag{2}$$

was originally considered in [11], where it was proved that it is not open convex and possesses a realization by a good open cover (Fig. 3b), thus does not have any "local obstructions" to convexity. However, it turns out that this code is closed convex (see a closed realization in Fig. 3a).

The examples in (1) and (2) show that open convex and closed convex are distinct classes of codes. Moreover, they illustrate that one cannot generally "convert" an open convex realization into a closed convex realization or vice versa by simply taking closures or interiors of sets in a cover: realizations of these codes by convex sets require fine-tuned intersections of boundaries, which results in a change in the code when closures or interiors are taken. Nevertheless, it is intuitive that open and closed versions of a "sufficiently non-degenerate" cover should yield the same code.

A natural candidate for such a condition would be that the sets in the cover $\mathcal{U}$ are *in general position*, i.e., there exists $\varepsilon > 0$ such that any cover $\mathcal{V} = \{V_i\}$ whose sets V_i are no further than ε from U_i in the Hausdorff distance,[7] has the same code: $\operatorname{code}(\mathcal{U}, \mathbb{R}^d) = \operatorname{code}(\mathcal{V}, \mathbb{R}^d)$. However, being in general position is too strong a condition; there are covers of interest that are not in general position, yet yield the same code after taking the closure or interior. For example, in the proof of Lemma 3.1, the "chipping away" process which introduces new codewords requires that the boundaries of the sets involved partially coincide in specified patterns. We wish to apply this result to construct both open and closed realizations in the proof of Theorem 1.2, so require the following weaker condition.

Definition 2.10 A cover $\mathcal{U} = \{U_i\}_{i\in[n]}$, with $U_i \subseteq \mathbb{R}^d$, is *non-degenerate* if the following two conditions hold:

[7] Recall that the Hausdorff distance between two subsets U and V of a Euclidean space is defined as

$$d_H(U, V) = \max\left\{\sup_{x\in U}\left\{\inf_{y\in V}\|x-y\|\right\}, \sup_{y\in V}\left\{\inf_{x\in U}\|x-y\|\right\}\right\}.$$

(i) For all $\sigma \in \text{code}(\mathcal{U}, \mathbb{R}^d)$, the atoms $A_\sigma^{\mathcal{U}}$ are *top-dimensional*, i.e., any non-empty intersection with an open set $B \subseteq \mathbb{R}^d$ has non-empty interior:

$$B \text{ is open and } A_\sigma^{\mathcal{U}} \cap B \neq \varnothing \quad \Longrightarrow \quad \text{int}(A_\sigma^{\mathcal{U}} \cap B) \neq \varnothing.$$

(ii) For all non-empty $\sigma \subseteq [n]$, $\bigcap_{i\in\sigma} \partial U_i \subseteq \partial\big(\bigcap_{i\in\sigma} U_i\big)$.

Note that if a cover $\mathcal{U}$ is open, convex and in general position, then it is non-degenerate (see Lemma A.3 in the Appendix), while an open convex and non-degenerate cover need not be in general position. On the other hand, a closed, convex cover in general position consisting of sets need not be non-degenerate: take any of the sets U_i to have empty interior. We should also note that the two seemingly separate conditions (i) and (ii) in Definition 2.10 are motivated by the following observation.

Lemma 2.11 *Assume that $\mathcal{U} = \{U_i\}$ is a finite cover by convex sets. Then*

(a) *if all U_i are open and $\mathcal{U}$ satisfies Definition* 2.10 (ii)*, then it also satisfies Definition* 2.10 (i)*;*
(b) *if all U_i are closed and $\mathcal{U}$ satisfies Definition* 2.10 (i)*, then it also satisfies Definition* 2.10 (ii).

The proof is given in the Appendix (Sect. A.2, Lemmas A.2 and A.4). Note that if the sets U_i are *open* and convex, then Definition 2.10 (i) does not imply Definition 2.10 (ii), similarly if the sets U_i are *closed* and convex then Definition 2.10 (ii) does not imply Definition 2.10 (i).[8]

For an open cover $\mathcal{U} = \{U_i\}$, we denote by $\text{cl}(\mathcal{U})$ the cover by the closures $V_i = \text{cl}(U_i)$. Similarly, for a closed cover $\mathcal{U} = \{U_i\}$ we denote by $\text{int}(\mathcal{U})$ the cover by the interiors $V_i = \text{int}(U_i)$. Recall that if a set is convex, then both its closure and its interior are convex.

Theorem 2.12 *Assume that $\mathcal{U} = \{U_i\}$ is a convex and non-degenerate cover, then*

$$U_i \text{ are open} \implies \text{code}(\mathcal{U}, \mathbb{R}^d) = \text{code}(cl(\mathcal{U}), \mathbb{R}^d);$$
$$U_i \text{ are closed} \implies \text{code}(\mathcal{U}, \mathbb{R}^d) = \text{code}(\text{int}(\mathcal{U}), \mathbb{R}^d).$$

That is, all codes which admit a realization by a convex, non-degenerate cover are both open and closed convex.

The proof is given in the Appendix (Sect. A.3). This theorem guarantees that if an open convex code is realizable by a non-degenerate cover, then it is also closed convex; similarly if a closed convex code is realizable by a non-degenerate cover, then it is also open convex. Non-degenerate covers are thus *natural* in the neuroscience context, where receptive fields (i.e., the sets U_i) should not change their code after taking closure or interior, since such changes in code would be undetectable in the presence of standard

[8] For example, the cover by the open convex sets $U_1 = \{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$ and $U_2 = \{(x, y) \in \mathbb{R}^2 \mid y < -x^2\}$ satisfies Definition 2.10 (i), but does not satisfy Definition 2.10 (ii). Similarly, the closed subsets of the real line, $U_1 = \{x \leq 0\}$, $U_2 = \{x \geq 0\}$ satisfy Definition 2.10 (ii), but do not satisfy Definition 2.10 (i).

neuronal noise. This suggests that convex codes that arise from non-degenerate covers should serve as the standard model for convex codes in neuroscience-related contexts. Note that the existence of a non-degenerate convex cover realization is *extrinsic* in that it is not defined in terms of the combinatorics of the code alone. A combinatorial description of such codes is unknown at the time of this writing.

3 Monotonicity of Open Convex Codes

The set of all codes $\mathcal{C} \subseteq 2^{[n]}$ with a prescribed simplicial complex $K = \Delta(C)$ forms a poset. It is easy to see that if $\mathcal{C}$ is an open or closed convex code then its sub-codes can be non-convex. For example any non-convex code is a sub-code of its simplicial complex, and every simplicial complex is both an open and closed convex code (this follows from Theorem 1.2 in Sect. 4). It turns out that open convexity is a monotone increasing property.

Theorem 1.3 *Assume that a code $\mathcal{C} \subset 2^{[n]}$ is open convex. Then every code $\mathcal{D}$ that satisfies $\mathcal{C} \subsetneq \mathcal{D} \subseteq \Delta(\mathcal{C})$ is also open convex with open embedding dimension* $\operatorname{odim} \mathcal{D} \leq \operatorname{odim} \mathcal{C} + 1$.

Note that the above bound on the embedding dimension is sharp. For example, the open convex code $\mathcal{C} = \{123, 12, 1\}$ has embedding dimension $\operatorname{odim} \mathcal{C} = 1$, but its simplicial complex $\mathcal{D} = \Delta(\mathcal{C})$ has embedding dimension $\operatorname{odim} \mathcal{D} = 2$. To prove this theorem we shall use the following lemma. Let $M(\mathcal{C})$ denote the facets[9] of the simplicial complex $\Delta(\mathcal{C})$.

Lemma 3.1 *Let $\mathcal{U} = \{U_i\}$ be an open convex cover in $\mathbb{R}^d$, $d \geq 2$, and $X \subseteq \mathbb{R}^d$ with $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$. Assume that there exists an open Euclidean ball $B \subset \mathbb{R}^d$ such that $\operatorname{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C}$, and for every $\alpha \in M(\mathcal{C})$, its atom has non-empty intersection with the boundary of B: $\partial B \cap A_\alpha^{\mathcal{U}} \neq \varnothing$. Then for every $\mathcal{D}$ such that $\mathcal{C} \subsetneq \mathcal{D} \subseteq \Delta(\mathcal{C})$, there exists an open convex cover $\mathcal{V} = \{V_i\}$ with $V_i \subseteq U_i$, such that $\mathcal{D} = \operatorname{code}(\mathcal{V}, B \cap X)$. Moreover, if the cover $\mathcal{U}$ is non-degenerate, then the cover $\mathcal{V}$ can also be chosen to be non-degenerate.*

The proof of this lemma is given in Sect. A.4. Intuitively, the reason why this lemma holds is that one can "chip away" small pieces from the ball B inside some atoms $A_\alpha^{\mathcal{U}}$ to uncover only the atoms corresponding to the codewords in $\mathcal{D} \backslash \mathcal{C}$.

Proof of Theorem 1.3 Assume that $\mathcal{U}$ is an open convex cover in $\mathbb{R}^d$ with $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$. Since there are only finitely many codewords, there exist a radius $r > 0$ and an open Euclidean ball $B_r^d \subset \mathbb{R}^d$, of radius r, centered at the origin, that satisfy $\operatorname{code}(\{B_r^d \cap U_i\}, B_r^d \cap X) = \mathcal{C}$. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be any linear projection. Let $B \stackrel{\text{def}}{=} B_r^{d+1}$ denote the open ball in $\mathbb{R}^{d+1}$, centered at the origin and of the same radius r. Define $\tilde{U}_i = \pi^{-1}(U_i)$. By construction, $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ is an open, convex cover, such that each of its atoms has non-empty intersection with the sphere ∂B. Moreover, $\operatorname{code}(\{B \cap \tilde{U}_i\}, B \cap \pi^{-1}(X)) = \mathcal{C}$. Thus the conditions of Lemma 3.1 are satisfied for the cover $\tilde{\mathcal{U}}$, and $\mathcal{D}$ is an open convex code with $\operatorname{odim} \mathcal{D} \leq \operatorname{odim} \mathcal{C} + 1$. ☐

[9] The *facets* of a simplicial complex those faces which are maximal under inclusion.

Note that the proof of Lemma 3.1 (see Sect. A.4) breaks down if one assumes that the convex sets U_i are closed. Moreover, it is currently not known if the monotonicity property holds in the setting of the closed convex codes. The differences between the open convex and the closed convex codes (described in the previous section) leave enough room for either possibility.

4 Max Intersection-Complete Codes are Open and Closed Convex

Here we introduce max intersection-complete codes and prove that they are open convex and closed convex. The open convexity of max intersection-complete codes was first hypothesized in [2].

Definition 4.1 The *intersection completion of a code* $\mathcal{C}$ is the code that consists of all non-empty intersections of codewords in $\mathcal{C}$:

$$\widehat{\mathcal{C}} = \left\{ \sigma \;\middle|\; \sigma = \bigcap_{\nu \in \mathcal{C}'} \nu \text{ for some non-empty subcode } \mathcal{C}' \subseteq \mathcal{C} \right\}.$$

Note that the intersection completion satisfies $\mathcal{C} \subseteq \widehat{\mathcal{C}} \subseteq \Delta(\mathcal{C})$.

Definition 4.2 Let $\mathcal{C} \subset 2^{[n]}$ be a code, and denote by $M(\mathcal{C}) \subset \mathcal{C}$ the subcode consisting of all maximal codewords[10] of $\mathcal{C}$. A code $\mathcal{C}$ is said to be

- *intersection-complete* if $\widehat{\mathcal{C}} = \mathcal{C}$;
- *max intersection-complete* if $\widehat{M(\mathcal{C})} \subseteq \mathcal{C}$.

Note that any simplicial complex (i.e., $\mathcal{C} = \Delta(\mathcal{C})$) is intersection-complete and any intersection-complete code is max intersection-complete. Intersection-complete codes allow a simple construction of a closed convex realization that we describe in Sect. A.5 (see Lemma A.9). However, in order to prove that max intersection-complete codes are both open and closed convex, we need the following.

Proposition 4.3 *Let $\mathcal{C} \subset 2^{[n]}$ be a code with k maximal elements. Then there exists an open convex and non-degenerate cover $\mathcal{U}$ in $d = (k-1)$-dimensional space whose code is the intersection completion of the maximal elements in $\mathcal{C}$:* $\text{code}(\mathcal{U}, \mathbb{R}^d) = \widehat{M(\mathcal{C})}$.

Proof Denote the maximal codewords as $M(\mathcal{C}) = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. If $k = 1$ this statement is trivially true. Assume $k \geq 2$ and consider a regular geometric $(k-1)$-simplex Δ^{k-1} in $\mathbb{R}^{k-1}$ constructed by evenly spacing vertices $[k]$ on the unit sphere $S^{k-2} \subseteq \mathbb{R}^{k-1}$. Construct a collection of hyperplanes $\{P_a\}_{a=1}^{k}$ in $\mathbb{R}^{k-1}$ by taking P_a to be the plane through the facet of $\partial \Delta^{k-1}$ which does not contain vertex a. Denote by H_a^+ the *closed* half-space containing the vertex a bounded by P_a and by H_a^- the complementary *open* half-space. Observe that this arrangement splits $\mathbb{R}^{k-1}$ into $2^k - 1$ disjoint, non-empty, convex chambers

[10] Equivalently, the facets of $\Delta(\mathcal{C})$.

$$H_\rho = \bigcap_{a\in\rho} H_a^+ \cap \bigcap_{b\notin\rho} H_b^-,$$

indexed by all non-empty[11] subsets $\rho \subseteq [k]$.

For every $i \in [n]$ consider $\rho(i) \overset{\text{def}}{=} \{a \in [k] \mid \sigma_a \ni i\} \subset [k]$, i.e., the collection of indices of the maximal codewords σ_a that contain i, and construct a collection of convex open sets $\mathcal{U} = \{U_i\}$

$$U_i \overset{\text{def}}{=} \coprod_{\rho\subseteq\rho(i)} H_\rho.$$

To show that the sets U_i are convex and open, observe that the above construction implies that we have the disjoint unions

$$\mathbb{R}^{k-1} = \coprod_{\rho\neq\varnothing} H_\rho \quad \text{and} \quad H_b^+ = \coprod_{\rho\ni b} H_\rho,$$

thus

$$\mathbb{R}^{k-1}\backslash U_i = \Big(\coprod_{\rho\neq\varnothing} H_\rho\Big) \setminus \Big(\coprod_{\rho\subseteq\rho(i)} H_\rho\Big) = \coprod_{\rho\nsubseteq\rho(i)} H_\rho = \bigcup_{b\notin\rho(i)} \Big(\coprod_{\rho\ni b} H_\rho\Big) = \bigcup_{b\notin\rho(i)} H_b^+.$$

Therefore, by de Morgan's Law,

$$U_i = \mathbb{R}^{k-1} \setminus \Big(\bigcup_{b\notin\rho(i)} H_b^+\Big) = \bigcap_{b\notin\rho(i)} H_b^-. \tag{3}$$

This is an intersection of open convex sets, and therefore open and convex. Note that if $\rho(i) = [k]$, this is an intersection over an empty index, and we interpret this set as all of $\mathbb{R}^{k-1}$.

To show that $\operatorname{code}(\mathcal{U}, \mathbb{R}^{k-1}) = \widehat{M(\mathcal{C})}$, observe that because the chambers of the hyperplane arrangement satisfy $H_\rho \cap H_\nu \neq \varnothing$ iff $\rho = \nu$, we have the following identities:

$$\bigcap_{i\in\sigma} U_i = \bigcap_{i\in\sigma} \coprod_{\rho\subseteq\rho(i)} H_\rho = \coprod_{\nu\subseteq\bigcap_{i\in\sigma}\rho(i)} H_\nu,$$
$$\bigcup_{j\notin\sigma} U_j = \bigcup_{j\notin\sigma} \coprod_{\nu\subseteq\rho(j)} H_\nu.$$

Thus, for a subset $\sigma \subseteq [n]$, the atom $A_\sigma^{\mathcal{U}}$ of the cover $\mathcal{U} = \{U_i\}$ is non-empty if and only if there exists a non-empty subset $\nu \subseteq [k]$ that satisfies the following two conditions:

$$\nu \subseteq \bigcap_{i\in\sigma} \rho(i); \tag{4}$$

11 The empty set is not included because under this definition, $H_\varnothing = \varnothing$.

$$\nu \not\subseteq \rho(j), \ \forall j \notin \sigma. \tag{5}$$

Note that the first condition (4) can be re-written as $\forall a \in \nu, \ \forall i \in \sigma, \ i \in \sigma_a$, which is equivalent to $\sigma \subseteq \bigcap_{a\in\nu} \sigma_a$. Similarly, the second condition (5) can be re-interpreted as

$$\forall j \notin \sigma, \ \exists b \in \nu, \ j \notin \sigma_b \iff [n]\backslash\sigma \subseteq \bigcup_{b\in\nu} [n]\backslash\sigma_b \iff \sigma \supseteq \bigcap_{b\in\nu} \sigma_b.$$

Thus $A_\sigma^{\mathcal{U}} \neq \varnothing \iff \sigma = \bigcap_{b\in\nu} \sigma_b$ for some non-empty $\nu \subseteq [k]$. This proves that $\mathrm{code}(\{U_i\}, \mathbb{R}^{k-1}) = \widehat{M(\mathcal{C})}$.

Lastly, we show that the cover $\mathcal{U}$ is non-degenerate. By construction, the half-spaces H_a^- are open, convex and in general position. Thus Lemma A.3 guarantees that the cover $\mathcal{H} = \{H_a^-\}$ is non-degenerate and using Lemma A.5 in the Appendix we conclude that for any non-empty $\tau \subseteq [k]$, $\bigcap_{a\in\tau} \mathrm{cl}(H_a^-) = \mathrm{cl}\big(\bigcap_{a\in\tau} H_a^-\big)$. For any non-empty subset $\sigma \subseteq [n]$ we can combine this with the equality (3) to obtain

$$\begin{aligned}\mathrm{cl}\Big(\bigcap_{i\in\sigma} U_i\Big) &= \mathrm{cl}\Big(\bigcap_{i\in\sigma} \bigcap_{a\notin\rho(i)} H_a^-\Big) = \bigcap_{i\in\sigma} \bigcap_{a\notin\rho(i)} \mathrm{cl}(H_a^-) \\ &= \bigcap_{i\in\sigma} \mathrm{cl}\Big(\bigcap_{a\notin\rho(i)} H_a^-\Big) = \bigcap_{i\in\sigma} \mathrm{cl}(U_i).\end{aligned}$$

Since U_i are open we obtain

$$\begin{aligned}\bigcap_{i\in\sigma} \partial U_i &= \bigcap_{i\in\sigma} \big(\mathrm{cl}(U_i) \setminus U_i\big) \subseteq \bigcap_{i\in\sigma} \Big(\mathrm{cl}(U_i) \setminus \bigcap_{i\in\sigma} U_i\Big) = \Big(\bigcap_{i\in\sigma} \mathrm{cl}(U_i)\Big) \setminus \bigcap_{i\in\sigma} U_i \\ &= \mathrm{cl}\Big(\bigcap_{i\in\sigma} U_i\Big) \setminus \bigcap_{i\in\sigma} U_i = \partial\Big(\bigcap_{i\in\sigma} U_i\Big).\end{aligned}$$

Therefore by Lemma 2.11 (a) the open and convex cover $\mathcal{U}$ is also non-degenerate. □

As a corollary we obtain the main result of this section:

Theorem 1.2 *Suppose $\mathcal{C} \subset 2^{[n]}$ is a max intersection-complete code. Then $\mathcal{C}$ is both open convex and closed convex with the embedding dimension $d \leq \max\{2, (k-1)\}$, where k is the number of facets of the complex $\Delta(\mathcal{C})$.*

Note that the case of $k = 1$, i.e., $M(\mathcal{C}) = \{[n]\}$, was proved in [2].

Proof We first consider the case when the number of maximal codewords is $k \geq 3$ and begin by constructing convex regions $\{H_\rho\}_{\rho\in 2^{[k]}\setminus\varnothing}$ and the open convex cover $\{U_i\}_{i=1}^n$ as in the proof of Proposition 4.3 (see Fig. 4). In this cover, every atom that corresponds to a maximal codeword is unbounded, therefore we can apply Lemma 3.1 using the open ball of radius 2 centered at the origin. This yields an open convex and non-degenerate cover, thus by Theorem 2.12 the code $\mathcal{C}$ is both open convex and closed convex.

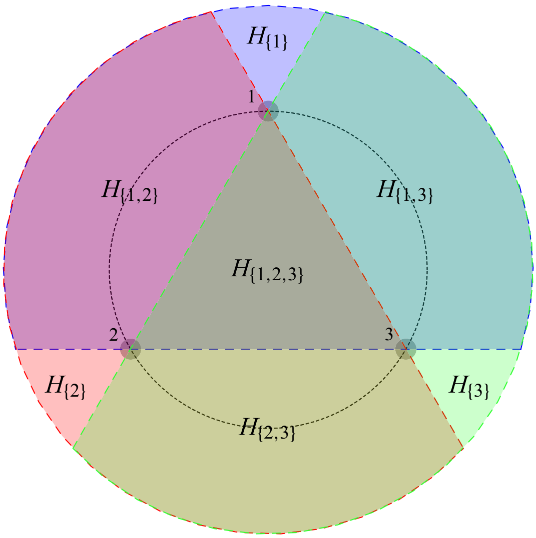


Fig. 4 The oriented hyperplane arrangement $\{P_a\}$ and its chambers H_ρ

For the case $k = 1$, we note that the proof given in [2] is equivalent to assigning each U_i to be the open ball of radius 2 centered at the origin in $\mathbb{R}^2$, and applying Lemma 3.1 to modify this cover to produce codewords properly contained in the unique maximal codeword. If $k = 2$, formally add $\gamma = \varnothing$ as an element of $M(\mathcal{C})$ and apply the same construction given above. As all intersections involving γ are empty, the sets U_i are contained entirely in H_γ^-, but no other modifications are necessary to perform the construction as described in the case of $k \geq 3$. □

Appendix A: Supporting Proofs

A.1 Proof of Lemma 2.9

Proof Consider the code $\mathcal{C}$ in (1) and assume that there exists a closed convex cover $\mathcal{U} = \{U_i\}$ in $\mathbb{R}^d$, with code$(\mathcal{U}, \mathbb{R}^d) = \mathcal{C}$. Without loss of generality, we can assume that the U_i are compact.[12] Because U_i are compact and convex one can pick points x_{123}, x_{345}, and x_{156} in the closed atoms $A^{\mathcal{U}}_{123}$, $A^{\mathcal{U}}_{345}$ and $A^{\mathcal{U}}_{156}$ respectively so that for every $a \in A^{\mathcal{U}}_{123}$, its distance to the closed line segment $M = \overline{x_{345}x_{156}}$ satisfies[13] $\text{dist}(a, M) \geq \text{dist}(x_{123}, M) \neq 0$, i.e., x_{123} minimizes the distance to the line segment M. Moreover, the points $x_{123}, x_{156}, x_{345}$ cannot be collinear. For the rest of this proof we will consider only the convex hull of these three points (Fig. 5).

[12] If U_i are not compact, then one can intersect them with a closed ball of large enough radius to obtain the same code.

[13] Because, U_5 is convex and contains the endpoints of M, $x_{123} \notin M$. Moreover, since both M and $A^{\mathcal{U}}_{123}$ are compact, the function $f(a) = \text{dist}(a, M)$ achieves its minimum on $A^{\mathcal{U}}_{123}$.

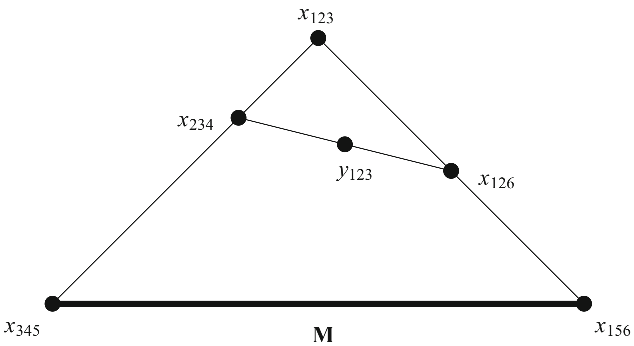


Fig. 5 The convex hull of x_{123}, x_{315}, and x_{156}. Points $x_{345} \in A^{\mathcal{U}}_{345}$ and $x_{156} \in A^{\mathcal{U}}_{156}$ can be arbitrarily chosen. Point $x_{123} \in A^{\mathcal{U}}_{123}$ is chosen to minimize the distance to the line segment M. A closer point $y_{123} \in A^{\mathcal{U}}_{123}$ is then constructed, providing the contradiction

Consider the closed line segment $L = \overline{x_{123}x_{156}}$. Because U_1 is convex, $L \subset U_1$, therefore the code (1) of the cover imposes that

$$L \subset A^{\mathcal{U}}_{123} \sqcup A^{\mathcal{U}}_{12} \sqcup A^{\mathcal{U}}_{126} \sqcup A^{\mathcal{U}}_{16} \sqcup A^{\mathcal{U}}_{156}.$$

Because each of the atoms above is contained in either U_2 or U_6, $L \subset U_2 \cup U_6$. Since L is connected and the sets $U_2 \cap L$ and $U_6 \cap L$ are closed and non-empty, we conclude that $U_2 \cap U_6 \cap L \subset A^{\mathcal{U}}_{126}$ must be non-empty, thus there exists a point $x_{126} \in A^{\mathcal{U}}_{126} \cap L$ that lies in the interior of L. By the same argument, there also exist points

$$x_{234} \in A^{\mathcal{U}}_{234} \text{ in the interior of } \overline{x_{123}x_{345}} \subset U_3, \text{ covered by } U_2 \text{ and } U_4,$$
$$y_{123} \in A^{\mathcal{U}}_{123} \text{ in the interior of } \overline{x_{234}x_{126}} \subset U_2, \text{ covered by } U_1 \text{ and } U_3,$$

and these points must lie on the interiors of their respective line segments (Fig. 5).

Finally we observe that because the point $y_{123} \in A^{\mathcal{U}}_{123}$ lies in the interior of a line segment $\overline{x_{234}x_{126}}$, it also lies in the interior of the closed triangle $\triangle(x_{123}, x_{156}, x_{345})$, and thus $d(y_{123}, M) < d(x_{123}, M)$. This yields a contradiction, since we chose $x_{123} \in A^{\mathcal{U}}_{123}$ to have the minimal distance to the line segment M. □

A.2 Proofs of Lemmas, Related to the Non-degeneracy Condition

We shall need the following several lemmas. The following lemma is well-known (see e.g., [7], exercises in Chapter 1), nevertheless we give its proof for the sake of completeness.

Lemma A.1 *For any finite cover $\mathcal{U} = \{U_i\}_{i=1}^n$ and a subset $\sigma \subseteq [n]$, the following hold:*

$$\mathrm{cl}\Big(\bigcup_{i\in\sigma} U_i\Big) = \bigcup_{i\in\sigma} \mathrm{cl}(U_i), \tag{6}$$

$$\mathrm{cl}\Big(\bigcap_{i\in\sigma} U_i\Big) \subseteq \bigcap_{i\in\sigma} \mathrm{cl}(U_i), \tag{7}$$

$$\mathrm{int}\Big(\bigcap_{i\in\sigma} U_i\Big) = \bigcap_{i\in\sigma} \mathrm{int}(U_i), \tag{8}$$

$$\mathrm{int}\Big(\bigcup_{i\in\sigma} U_i\Big) \supseteq \bigcup_{i\in\sigma} \mathrm{int}(U_i). \tag{9}$$

Proof Observe that since $U_i \subseteq \mathrm{cl}(U_i)$, we have $\bigcup_{i\in\sigma} U_i \subseteq \bigcup_{i\in\sigma} \mathrm{cl}(U_i)$ and thus

$$\mathrm{cl}(\bigcup_{i\in\sigma} U_i) \subseteq \mathrm{cl}\Big(\bigcup_{i\in\sigma} \mathrm{cl}(U_i)\Big) = \bigcup_{i\in\sigma} \mathrm{cl}(U_i). \tag{10}$$

Similarly, we find the inclusion (7). Using $U_i \supseteq \mathrm{int}(U_i)$, one also obtains the inclusion (9) and the inclusion

$$\mathrm{int}\Big(\bigcap_{i\in\sigma} U_i\Big) \supseteq \bigcap_{i\in\sigma} \mathrm{int}(U_i). \tag{11}$$

Observe that for any $j \in \sigma$, $\mathrm{cl}(U_j) \subseteq \mathrm{cl}\big(\bigcup_{i\in\sigma} U_i\big)$ and $\mathrm{int}(U_j) \supseteq \mathrm{int}\big(\bigcap_{i\in\sigma} U_i\big)$, thus we obtain $\bigcup_{i\in\sigma} \mathrm{cl}(U_i) \subseteq \mathrm{cl}\big(\bigcup_{i\in\sigma} U_i\big)$ and $\bigcap_{i\in\sigma} \mathrm{int}(U_i) \supseteq \mathrm{int}\big(\bigcap_{i\in\sigma} U_i\big)$. These combined with (10) and (11) yield (6) and (8) respectively. □

Lemma A.2 (Lemma 2.11(a)) *Suppose $\mathcal{U} = \{U_i\}_{i\in[n]}$ is an open and convex cover such that for every non-empty subset $\tau \subseteq [n]$, $\bigcap_{i\in\tau} \partial U_i \subseteq \partial\big(\bigcap_{i\in\tau} U_i\big)$. Then every atom of $\mathcal{U}$ is top-dimensional.*

Proof Assume the converse, i.e., there exists non-empty $\sigma \subset [n]$ and an open subset $B \subseteq \mathbb{R}^d$ such that that $A_\sigma^{\mathcal{U}} \cap B \neq \varnothing$ and $\mathrm{int}(A_\sigma^{\mathcal{U}} \cap B) = \varnothing$. Let $x \in A_\sigma^{\mathcal{U}} \cap B$, and denote by $\tau \subset [n]\backslash\sigma$, the maximal subset such that $x \in \bigcap_{j\in\tau} \partial U_j$. Note that τ is non-empty[14] and therefore (using the assumption of the lemma) $x \in \partial\big(\bigcap_{j\in\tau} U_j\big)$. Denote by $\varepsilon_0 > 0$ the maximal radius such that the open ball $B_{\varepsilon_0}(x)$ satisfies (a) $B_{\varepsilon_0}(x) \subset B \cap \bigcap_{i\in\sigma} U_i$ and (b) for every $l \notin (\sigma \cup \tau)$, $B_{\varepsilon_0}(x) \cap U_l = \varnothing$.

Observe that for every point $y \in \bigcap_{j\in\tau} U_j$ and every $\varepsilon \in (0, \varepsilon_0)$, the point $z_\varepsilon(y) = x + \varepsilon \frac{x-y}{\|x-y\|}$ satisfies $z_\varepsilon(y) \notin U_j$ for every $j \notin \sigma$. This is because for every $j \in \tau$, the open set U_j is convex, thus if $x \in \partial U_j$, and $y \in U_j$, then $z_\varepsilon(y) \notin U_j$, as in

[14] If $x \notin \partial U_j\ \forall j \notin \sigma$, then (because U_i are open) there exists a small open ball $B' \ni x$ such that $B' \subset A_\sigma^{\mathcal{U}}$, thus $\mathrm{int}(A_\sigma^{\mathcal{U}} \cap B) \supseteq \mathrm{int}(A_\sigma^{\mathcal{U}} \cap B \cap B') \neq \varnothing$, a contradiction.

Fig. 6 Construction of points in $\mathrm{int}(A^{\mathcal{U}}_{\sigma} \cap B)$ from the proof of Lemma A.2

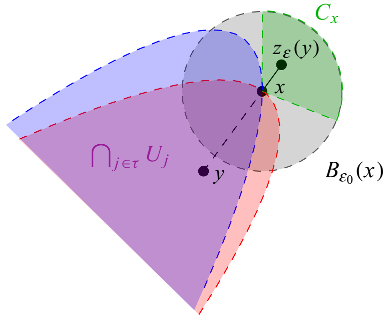


Fig. 6. We thus conclude that $z_\varepsilon(y) \in A^{\mathcal{U}}_{\sigma} \cap B$. Since the intersection $\bigcap_{j\in\tau} U_j$ is open, the totality of all such points $z_\varepsilon(y)$ form an open cone $C_x \subseteq A^{\mathcal{U}}_{\sigma} \cap B$. Therefore $\mathrm{int}(A^{\mathcal{U}}_{\sigma} \cap B) \supseteq \mathrm{int}(C_x) \neq \varnothing$, a contradiction. □

Lemma A.3 *Suppose $\mathcal{U}$ is an open and convex cover in general position, then $\mathcal{U}$ is a non-degenerate cover.*

Proof Assume $\mathcal{U}$ is in general position, open, convex, yet *not* non-degenerate. Then, by Lemma A.2 there exists a non-empty subset $\sigma \subseteq [n]$ so that $\bigcap_{i\in\sigma} \partial U_i \not\subseteq \partial\big(\bigcap_{i\in\sigma} U_i\big)$. Let us choose $x \in \big(\bigcap_{i\in\sigma} \partial U_i\big)\backslash\big(\partial \bigcap_{i\in\sigma} U_i\big)$. Suppose there exists $z \in \bigcap_{i\in\sigma} U_i$, then the open line segment between x and z is contained in $\bigcap_{i\in\sigma} U_i$, and thus $x \in \partial\big(\bigcap_{i\in\sigma} U_i\big)$, a contradiction. Therefore, $\bigcap_{i\in\sigma} U_i = \varnothing$, and for every $\tau \supseteq \sigma$, $\tau \notin \mathrm{code}(\mathcal{U}, \mathbb{R}^d)$.

For any $\varepsilon > 0$, define an open cover $\mathcal{V}(\varepsilon) = \{V_i(\varepsilon)\}$ by $V_i(\varepsilon) = U_i \cup B_\varepsilon(x)$ for $i \in \sigma$ and $V_j(\varepsilon) = U_j$ otherwise. Notice that $\bigcap_{i\in\sigma} V_i(\varepsilon) = B_\varepsilon(x)$. Thus for any $\varepsilon > 0$, there exists $\tau \supseteq \sigma$ with $\tau \in \mathrm{code}(\mathcal{V}(\varepsilon), \mathbb{R})$. Because x lies in the boundary of U_i for each $i \in \sigma$, each $V_i(\varepsilon)$ is no more than ε away from U_i w.r.t. the Hausdorff distance. Therefore $\mathcal{U}$ is not in general position, a contradiction. □

Lemma A.4 (Lemma 2.11(b)) *Assume that every atom of the cover $\mathcal{U} = \{U_i\}$ is top-dimensional, i.e., any non-empty intersection with an open set $B \subseteq \mathbb{R}^d$ has non-empty interior, and the subsets U_i are closed and convex, then for any non-empty $\tau \subseteq [n]$,*

$$\bigcap_{i\in\tau} \partial U_i \subseteq \partial\Big(\bigcup_{i\in\tau} U_i\Big), \tag{12}$$

$$\bigcap_{i\in\tau} \partial U_i \subseteq \partial\Big(\bigcap_{i\in\tau} U_i\Big). \tag{13}$$

Proof To show (12) assume the converse. Then there exist a point $x \in \big(\bigcap_{i\in\tau} \partial U_i\big) \cap \mathrm{int}\big(\bigcup_{i\in\tau} U_i\big)$ at the interior, and an open ball $B \ni x$, such that $B \subseteq \bigcup_{i\in\tau} U_i$. First, let us show that these assumptions imply that

$$B \cap \bigcap_{i\in\tau} \mathrm{int}(U_i) = \varnothing. \tag{14}$$

Indeed, if there existed a point $y \in B \cap \bigcap_{i\in\tau} \operatorname{int}(U_i)$, then for every $\varepsilon > 0$ such that $z = x + \varepsilon(x - y) \in B$ and every $i \in \tau$, $z \notin U_i$ by convexity of U_i.[15] This implies $B \not\subseteq \bigcup_{i\in\tau} U_i$, a contradiction, thus (14) holds.

Denote by $\rho \supseteq \tau$ the element of $\operatorname{code}(\{U_i\}, \mathbb{R}^d)$ such that $x \in A^{\mathcal{U}}_\rho = \bigcap_{i\in\rho} U_i \setminus \bigcup_{j\notin\rho} U_j$. Because the sets U_j are closed, we can choose the open ball $B \ni x$, that satisfies (14) so that it is disjoint from $\bigcup_{j\notin\rho} U_j$. Therefore, using (8), we obtain

$$\operatorname{int}(B \cap A^{\mathcal{U}}_\rho) = \operatorname{int}\left(B \cap \bigcap_{i\in\rho} U_i\right) \subseteq \operatorname{int}\left(B \cap \bigcap_{i\in\tau} U_i\right) = B \cap \bigcap_{i\in\tau} \operatorname{int}(U_i) = \varnothing.$$

Since $x \in B \cap A^{\mathcal{U}}_\rho$, this contradicts the non-degeneracy of $\mathcal{U}$, and thus finishes the proof of (12).

To prove (13), consider $x \in \bigcap_{i\in\tau} \partial U_i \subseteq \bigcap_{i\in\tau} U_i$. Because of (12), any open neighborhood $O \ni x$ satisfies $O \not\subseteq \bigcup_{i\in\tau} U_i$ and thus $O \not\subseteq \bigcap_{i\in\tau} U_i$. Therefore $x \in \partial\big(\bigcap_{i\in\tau} U_i\big)$. ☐

Note that if the condition that the sets U_i are convex is violated, then the conclusions of the above lemma may not hold. For example, the sets $U_1 = \{(x, y) \in \mathbb{R}^2 \mid y \le x^2\}$ and $U_2 = \{(x, y) \in \mathbb{R}^2 \mid y \ge -x^2\}$ do not satisfy the inclusion (12).

Lemma A.5 *If the cover $\mathcal{U} = \{U_i\}_{i\in[n]}$ is non-degenerate, then for every non-empty subset $\sigma \subseteq [n]$*

$$U_i \text{ are closed and convex} \implies \operatorname{int}\left(\bigcup_{i\in\sigma} U_i\right) = \bigcup_{i\in\sigma} \operatorname{int}(U_i), \tag{15}$$

$$U_i \text{ are open and convex} \implies \operatorname{cl}\left(\bigcap_{i\in\sigma} U_i\right) = \bigcap_{i\in\sigma} \operatorname{cl}(U_i). \tag{16}$$

Proof First, we show that if the cover $\mathcal{U}$ is non-degenerate and closed convex, then

$$\operatorname{int}\left(\bigcup_{i\in\sigma} U_i\right) \subseteq \bigcup_{i\in\sigma} \operatorname{int}(U_i). \tag{17}$$

It suffices to show that if $x \notin \bigcup_{i\in\sigma} \operatorname{int}(U_i)$, then $x \in \partial\big(\bigcup_{i\in\sigma} U_i\big) \cup \big(\mathbb{R}^d \setminus \bigcup_{i\in\sigma} U_i\big)$. If $x \notin \bigcup_{i\in\sigma} U_i$, then this is true, thus we can assume that the set $\tau \overset{\text{def}}{=} \{i \in \sigma \mid x \in U_i\}$ is non-empty, and since $x \notin \bigcup_{i\in\sigma} \operatorname{int}(U_i)$, we conclude that $x \in \bigcap_{i\in\tau} \partial U_i$. Thus by Lemma A.4 (12), $x \in \partial\big(\bigcup_{i\in\tau} U_i\big)$. Now observe that $\bigcup_{i\in\sigma} U_i = A \cup B$ with $A \overset{\text{def}}{=} \bigcup_{i\in\tau} U_i$ and $B \overset{\text{def}}{=} \bigcup_{j\in\sigma\setminus\tau} U_j$. Since $x \notin B$, and B is closed, there exists an open neighborhood $O \ni x$ with $O \cap B = \varnothing$. Therefore, using (8) we obtain that

$$O \cap \operatorname{int}(A) = \operatorname{int}(O \cap A) = \operatorname{int}(O \cap (A \cup B)) = O \cap \operatorname{int}(A \cup B),$$

[15] This is because $y \in \operatorname{int}(U_i)$, $x \in \partial U_i$ and U_i is convex, thus for every $\varepsilon > 0$, $z = x + \varepsilon(x - y) \notin U_i$.

and thus we conclude

$$x \in \partial A \cap O = (A \setminus \operatorname{int} A) \cap O = ((A \cup B) \setminus (\operatorname{int}(A \cup B) \cap O)) \cap O$$
$$= \partial(A \cup B) \cap O.$$

Thus, $x \in \partial(A \cup B) = \partial\big(\bigcup_{i \in \sigma} U_i\big)$, which proves (17). Combined with (9) in Lemma A.1, this finishes the proof of (15).

To prove (16), taking into account (7), we need to show that $\operatorname{cl}\big(\bigcap_{i \in \sigma} U_i\big) \supseteq \bigcap_{i \in \sigma} \operatorname{cl}(U_i)$. Assume the converse, then there exists $x \in \bigcap_{i \in \sigma} \operatorname{cl}(U_i)$ and $r > 0$ such that

$$\forall \varepsilon \in (0, r) \text{ the open } \varepsilon\text{-ball } B_\varepsilon(x) \text{ satisfies } B_\varepsilon(x) \cap \bigcap_{i \in \sigma} U_i = \varnothing. \tag{18}$$

Denote $\tau \stackrel{\text{def}}{=} \{i \in \sigma \mid x \in \partial U_i\}$; we can assume that τ is non-empty (otherwise, $x \in \operatorname{cl}\big(\bigcap_{i \in \sigma} U_i\big)$). Using the condition (ii) of Definition 2.10 we conclude $x \in \bigcap_{i \in \tau} \partial U_i \subseteq \partial\big(\bigcap_{i \in \tau} U_i\big)$, thus for every open ε-ball $B_\varepsilon(x)$ centered at x, $B_\varepsilon(x) \cap \bigcap_{i \in \tau} U_i \neq \varnothing$. Because $x \in \bigcap_{j \in \sigma \setminus \tau} U_j$, and U_j are open, for a sufficiently small ε, $B_\varepsilon(x) \subset \bigcap_{j \in \sigma \setminus \tau} U_j$. Thus $B_\varepsilon(x) \cap \bigcap_{i \in \sigma} U_i \neq \varnothing$, which contradicts (18). This finishes the proof of (16). □

A.3 Proof of Theorem 2.12

Proof First, we show that if $\mathcal{U}$ is open convex and non-degenerate, then the cover of closures $\operatorname{cl}(\mathcal{U}) \stackrel{\text{def}}{=} \{\operatorname{cl}(U_i)\}$ has the same code as $\mathcal{U}$. Let $A_\sigma^{\mathcal{U}}$ denote an atom of $\mathcal{U}$ and $A_\sigma^{\operatorname{cl}(\mathcal{U})}$ denote the corresponding atom of $\operatorname{cl}(\mathcal{U})$. If $A_\sigma^{\mathcal{U}} = \varnothing$, then using (16) and (6) we conclude that

$$\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \notin \sigma} U_j \implies \operatorname{cl}\Big(\bigcap_{i \in \sigma} U_i\Big) \subseteq \operatorname{cl}\Big(\bigcup_{j \notin \sigma} U_j\Big) \implies \bigcap_{i \in \sigma} \operatorname{cl}(U_i) \subseteq \bigcup_{j \notin \sigma} \operatorname{cl}(U_j),$$

and thus $A_\sigma^{\operatorname{cl}(\mathcal{U})} = \varnothing$. Therefore, $\operatorname{code}(\operatorname{cl}(\mathcal{U})) \subseteq \operatorname{code}(\mathcal{U})$. On the other hand, using (6) we obtain

$$A_\sigma^{\operatorname{cl}(\mathcal{U})} = \bigcap_{i \in \sigma} \operatorname{cl}(U_i) \setminus \bigcup_{j \notin \sigma} \operatorname{cl}(U_j) = \bigcap_{i \in \sigma} \operatorname{cl}(U_i) \setminus \operatorname{cl}\Big(\bigcup_{j \notin \sigma} U_j\Big)$$
$$= \Big(\bigcap_{i \in \sigma} \operatorname{cl}(U_i) \setminus \bigcup_{j \notin \sigma} U_j\Big) \setminus \Big(\operatorname{cl}\Big(\bigcup_{j \notin \sigma} U_j\Big) \setminus \bigcup_{j \notin \sigma} U_j\Big) \supseteq A_\sigma^{\mathcal{U}} \setminus \partial\Big(\bigcup_{j \notin \sigma} U_j\Big).$$

Thus, if $A_\sigma^{\mathcal{U}}$ is non-empty, since it is top-dimensional while $\partial\big(\bigcup_{i \notin \sigma} U_i\big)$ is of codimension one, $A_\sigma^{\mathcal{U}} \not\subseteq \partial\big(\bigcup_{j \notin \sigma} U_j\big)$, implying $A_\sigma^{\operatorname{cl}(\mathcal{U})} \neq \varnothing$, and thus, $\operatorname{code}(\mathcal{U}) = \operatorname{code}(\operatorname{cl}(\mathcal{U}))$.

Next, we show that if $\mathcal{U}$ is closed convex and non-degenerate, then the cover of the interiors $\mathrm{int}(\mathcal{U}) = \{\mathrm{int}\, U_j\}$ has the same code as U. Let $A^{\mathcal{U}}_\sigma$ be an atom of $\mathcal{U}$ and $A^{\mathrm{int}(\mathcal{U})}_\sigma$ be the corresponding atom of $\mathrm{int}(\mathcal{U})$. If $A^{\mathcal{U}}_\sigma = \varnothing$, then using (8) and (15) we conclude that

$$\bigcap_{i\in\sigma} U_i \subseteq \bigcup_{j\notin\sigma} U_j \quad\Longrightarrow\quad \mathrm{int}\Big(\bigcap_{i\in\sigma} U_i\Big) \subseteq \mathrm{int}\Big(\bigcup_{j\notin\sigma} U_j\Big)$$
$$\Longrightarrow\quad \bigcap_{i\in\sigma} \mathrm{int}(U_i) \subseteq \bigcup_{j\notin\sigma} \mathrm{int}(U_j),$$

which implies $A^{\mathrm{int}(\mathcal{U})}_\sigma = \varnothing$. Therefore, $\mathrm{code}(\mathrm{int}(\mathcal{U})) \subseteq \mathrm{code}(\mathcal{U})$. On the other hand, using (8) we obtain

$$A^{\mathrm{int}(\mathcal{U})}_\sigma = \bigcap_{i\in\sigma} \mathrm{int}(U_i) \setminus \bigcup_{j\notin\sigma} \mathrm{int}(U_j) \supset \mathrm{int}\Big(\bigcap_{i\in\sigma} U_i\Big) \setminus \bigcup_{j\notin\sigma} U_j$$
$$= \Big(\bigcap_{i\in\sigma} U_i \setminus \bigcup_{j\notin\sigma} U_j\Big) \setminus \partial\Big(\bigcap_{i\in\sigma} U_i\Big) = A^{\mathcal{U}}_\sigma \setminus \partial\Big(\bigcap_{i\in\sigma} U_i\Big).$$

Thus, if $A^{\mathcal{U}}_\sigma$ is non-empty, since it is top-dimensional while $\partial\big(\bigcap_{i\in\sigma} A_i\big)$ is of codimension one, $A^{\mathrm{int}(\mathcal{U})}_\sigma \neq \varnothing$. Therefore, $\mathrm{code}(\mathcal{U}) = \mathrm{code}(\mathrm{int}(\mathcal{U}))$. □

A.4 Proof of Lemma 3.1

In order to prove Lemma 3.1 we will need the following two lemmas.

Lemma A.6 *Let $\mathcal{W} = \{W_i\}$ be a collection of sets, $W_i \subseteq X$, and $\mathcal{C} = \mathrm{code}(\mathcal{W}, X)$. Assume that Q is a proper subset of some atom of $\mathcal{W}$, i.e., $\varnothing \neq Q \subsetneq A^{\mathcal{W}}_\alpha$, for a non-empty $\alpha \in \mathcal{C}$. Then for any $\sigma_0 \subsetneq \alpha$, the cover $\mathcal{V} = \{V_i\}$ by the sets*

$$V_i = \begin{cases} W_i, & \text{if } i \in \sigma_0, \\ W_i \setminus Q, & \text{if } i \notin \sigma_0 \end{cases} \tag{19}$$

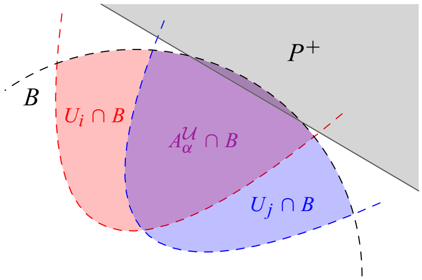
adds the codeword σ_0 to the original code, i.e., $\mathrm{code}(\mathcal{V}, X) = \mathrm{code}(\mathcal{W}, X) \cup \{\sigma_0\}$.

Proof Since $Q \subsetneq A^{\mathcal{W}}_\alpha$, $\mathrm{code}(\{V_i \cap (X\setminus Q)\}, X\setminus Q) = \mathrm{code}(\mathcal{W}, X)$. Moreover, because $\sigma_0 \subset \alpha$, $\mathrm{code}(\{V_i \cap Q\}, Q) = \{\sigma_0\}$ by construction. Finally, observe that if $X = Y \sqcup Z$, then $\mathrm{code}(\mathcal{V}, X) = \mathrm{code}(\{V_i \cap Y\}, Y) \cup \mathrm{code}(\{V_i \cap Z\}, Z)$, therefore we obtain

$$\mathrm{code}(\mathcal{V}, X) = \mathrm{code}(\{V_i \cap (X\setminus Q)\}, X\setminus Q) \cup \mathrm{code}(\{V_i \cap Q\}, Q) = \mathrm{code}(\mathcal{W}, X) \cup \{\sigma_0\}.$$

□

Fig. 7 The oriented half space P^+ is chosen to intersect the ball B inside $A_\alpha^{\mathcal{U}}$



Recall that $M(\mathcal{C}) \subset \mathcal{C}$ denotes the set of maximal codewords of $\mathcal{C}$. A subset $A \cap B$ of a topological space is called *relatively open in* B if it is an open set in the induced topology of the subset B.

Lemma A.7 *Let* $\mathcal{U} = \{U_i\}$ *be an open convex cover in* $\mathbb{R}^d$, $d \geq 2$, *and* $X \subseteq \mathbb{R}^d$ *with* $\mathcal{C} = \operatorname{code}(\mathcal{U}, X)$. *Assume that there exists an open Euclidean ball* $B \subset \mathbb{R}^d$ *such that* $\operatorname{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C}$, *and for every maximal set* $\alpha \in M(\mathcal{C})$, *the set* $\partial B \cap \operatorname{cl}\left(\bigcap_{i\in\alpha} U_i\right)$ *is non-empty and is relatively open in* ∂B. *Then for any simplicial violator* $\sigma_0 \in \Delta(\mathcal{C})\backslash\mathcal{C}$, *there exists an open convex cover* $\mathcal{V} = \{V_i\}$ *with* $V_i \subseteq U_i$, *so that* $\operatorname{code}(\mathcal{V}, B \cap X) = \mathcal{C} \cup \sigma_0$, *and the cover* $\mathcal{V}$ *satisfies the same condition above with the same open ball* B. *Moreover, if the cover* $\mathcal{U} = \{U_i\}$ *is non-degenerate, then the cover* $\mathcal{V}$ *can also be chosen to be non-degenerate.*

Proof Choose a facet $\alpha \in M(\mathcal{C})$ such that $\alpha \supsetneq \sigma_0$. Because α is a facet of $\Delta(\mathcal{C})$, the atom $A_\alpha^{\mathcal{U}} = \bigcap_{i\in\alpha} U_i$ is convex open and (by assumption) has a non-empty relatively open intersection with the Euclidean sphere ∂B. This implies that we can always select an oriented and *closed* half-space $P^+ \subset \mathbb{R}^d$ such that $P^+ \cap B \subset A_\alpha^{\mathcal{U}}$, and $(A_\alpha^{\mathcal{U}} \cap B)\backslash P^+ \neq \varnothing$ has relatively open intersection with the sphere ∂B (see Fig. 7).

We define two open covers, $\mathcal{W} = \{W_i\}$, with $W_i \stackrel{\text{def}}{=} U_i \cap B$ and $\mathcal{V} = \{V_i\}$ via (19), with $Q = P^+ \cap B$. We thus can use Lemma A.6, and conclude that $\operatorname{code}(\mathcal{V}, X \cap B) = \operatorname{code}(\{B \cap U_i\}, B \cap X) \cup \sigma_0 = \mathcal{C} \cup \sigma_0$. Note that by construction the sets V_i are open and convex, moreover, the cover $\mathcal{V}$ automatically satisfies the same condition on the atoms of facets of $\Delta(\mathcal{C})$.

Finally, if $\mathcal{U}$ is non-degenerate, then $\mathcal{V}$ is also non-degenerate. Indeed, because $A_\alpha^{\mathcal{U}}$ is open, the only two atoms that were changed, $A_\alpha^{\mathcal{V}} = (A_\alpha^{\mathcal{U}} \cap B)\backslash P^+$ and $A_{\sigma_0}^{\mathcal{V}} = P^+ \cap B$ are also top-dimensional. Moreover, since the only new pieces of boundaries of $V_i \subseteq U_i$ are introduced on the chord $\partial P^+ \cap B$ and on the sphere ∂B, if the condition that for all $\sigma \subseteq [n]$, $\bigcap_{i\in\sigma} \partial U_i \subseteq \partial\left(\bigcap_{i\in\sigma} U_i\right)$ holds then the same condition should hold for the sets V_i. □

A consecutive application of the above lemma to all the codewords in $\mathcal{D}\backslash\mathcal{C}$ for any supra-code $\mathcal{D}$ with the same simplicial complex yields Lemma 3.1.

Proof of Lemma 3.1 Let $\mathcal{U} = \{U_i\}$ be an open convex cover in $\mathbb{R}^d$, $d \geq 2$, and $X \subseteq \mathbb{R}^d$ with $\operatorname{code}(\mathcal{U}, X) = \mathcal{C}$. Assume that there exists an open Euclidean ball $B \subset \mathbb{R}^d$ such that $\operatorname{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C}$, and for every $\alpha \in M(\mathcal{C})$, its atom has non-empty intersection with ∂B. Let $\mathcal{C} \subsetneq \mathcal{D} \subseteq \Delta(\mathcal{C})$ and denote $\mathcal{D} \backslash \mathcal{C} = \{\sigma_1, \sigma_2, \dots, \sigma_l\}$. Let $\sigma_1 \subsetneq \alpha \in M(\mathcal{C})$. Because $\alpha \in M(C)$, $A_\alpha^{\mathcal{U}} = \cap_{i \in \alpha} U_i$ is open, and thus $\partial B \cap \mathrm{cl}(A_\alpha^{\mathcal{U}})$ is relatively open in ∂B. We can now apply Lemma A.7 to the "missing" codeword σ_1, and obtain a new cover $\mathcal{V}^{(1)}$ that again satisfies the condition of Lemma A.7. Consecutively applying Lemma A.7 with $\sigma_0 = \sigma_j$, $j = 2, 3, \dots, l$, we obtain covers $\mathcal{V}^{(j)}$, so that the last cover, $\mathcal{V} \stackrel{\text{def}}{=} \mathcal{V}^{(l)}$ is the desired cover of Lemma 3.1. □

A.5 A Closed Convex Realization for an Intersection-Complete Code

Here we provide an explicit construction of a closed convex cover of an intersection-complete code. Intersection-complete codes are max intersection-complete, and thus Theorem 1.2 ensures that intersection-complete codes are both open convex and closed convex. Nevertheless, a different construction below may be useful for applications due to its simplicity.

Definition A.8 The *potential cover* of the code $\mathcal{C}$, is a collection $\mathcal{V} = \{V_i\}_{i \in [n]}$ of closed convex sets $V_i \subset \mathbb{R}^{|\mathcal{C}|}$, defined as follows. For each non-empty codeword $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$. For each $i \in [n]$, we define V_i as the convex hull

$$V_i \stackrel{\text{def}}{=} \mathrm{conv}\{e_\sigma \mid \sigma \in \mathcal{C},\ \sigma \ni i\}.$$

Since this is a cover by convex closed sets, the code of the potential cover is closed convex. Note however, that this cover is *not* non-degenerate (Definition 2.10), and cannot be easily extended to an open convex cover.

Lemma A.9 *Let* $\mathcal{V} = \{V_i\}$ *denote the potential cover of* $\mathcal{C}$*, and* $X \stackrel{\text{def}}{=} \mathrm{conv}\{e_\sigma \mid \sigma \in \mathcal{C},\ \sigma \neq \varnothing\}$. *Then the code of the potential cover of* $\mathcal{C}$ *is the intersection completion of that code:* $\mathrm{code}(\mathcal{V}, X) = \widehat{\mathcal{C}}$.

Proof Note that because the vectors e_σ are linearly independent,

$$\varnothing \notin \widehat{\mathcal{C}} \iff \exists\, i \in [n],\ V_i = X \iff X = \bigcup_{i \in [n]} V_i \iff \varnothing \notin \mathrm{code}(\mathcal{V}, X).$$

Moreover,

$$\bigcap_{i \in \sigma} V_i = \mathrm{conv}\{e_\tau \mid \tau \in \mathcal{C},\ \tau \supseteq \sigma\}, \tag{20}$$

in particular, $\mathrm{code}(\mathcal{V}, X) \subseteq \Delta(\mathcal{C})$. To show that $\mathrm{code}(\mathcal{V}, X) \subseteq \widehat{\mathcal{C}}$, assume that a non-empty $\sigma \in \mathrm{code}(\mathcal{V}, X)$, i.e., $A_\sigma^{\mathcal{V}} = \left(\bigcap_{i \in \sigma} V_i\right) \backslash \bigcup_{j \notin \sigma} V_j$ is non-empty. If there exists an index $j \in \left(\bigcap_{\sigma \subseteq \tau \in \mathcal{C}} \tau\right) \backslash \sigma$, then by (20), $\bigcap_{i \in \sigma} V_i \subset V_j$, which contradicts $\sigma \in \mathrm{code}(\mathcal{V}, X)$. Hence $\sigma = \bigcap_{\mathcal{C} \ni \tau \supseteq \sigma} \tau \in \widehat{\mathcal{C}}$. Conversely, assume that a non-empty $\sigma \in \widehat{\mathcal{C}}$ and let $\sigma_1, \dots, \sigma_k \in \mathcal{C}$ be code elements such that $\sigma = \bigcap_{\ell=1}^k \sigma_\ell$. Then the point $\frac{1}{k} \sum_{\ell=1}^k e_{\sigma_\ell} \in \left(\bigcap_{i \in \sigma} V_i\right) \backslash \bigcup_{j \notin \sigma} V_j$. Hence $\sigma \in \mathrm{code}(\mathcal{V}, X)$. □

An immediate corollary is that any intersection-complete code is a closed convex code.

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