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Infinitesimal Poisson Cohomology

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ABSTRACT. The cohomology spaces arising in the infinitesimal Poisson geometry are considered in the Lie algebroids framework. It is shown that Lie algebroid structure can be reduced to the vector bundle over an integral leaf of the anchor image. In particular, the Schouten–Lichnerowicz complex can be reduced to a complex associated with the normal vector bundle over each symplectic leaf. The reduced Lie algebroid is transitive and its cohomology is finite-dimensional.

The calculation of cohomology with trivial coefficients for transitive Lie algebroids is carried out following the general approach due to Mackenzie. Simple formulas are given in some particular cases. In the case of Abelian algebroids, a classification theorem and formulas for some cohomology spaces are given.

A formal equivalence problem is considered for two Poisson brackets having the same symplectic leaf. Sufficient conditions for the formal equivalence are given in terms of certain cohomology spaces of a transitive Lie algebroid.

Contents

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Introduction

The Poisson cohomology introduced by A. Lichnerowicz [$\mathbf{Li}_2$] occurs in many problems of Hamiltonian mechanics and quantization [**BFFLS, Hu, KM,** $\mathbf{Li}_2$, **Va,** $\mathbf{VK}_{1,2}$]. For regular Poisson manifolds, the method for computation of this cohomology was first developed in [$\mathbf{VK}_{1,2}$]; see also [**KM, Va, Xu**]. In the case of an action of a compact Poisson group, the Poisson cohomology was computed in [**Gi**]. The problem of calculating the Poisson cohomology for general (irregular) Poisson manifolds seems to be very attractive but rather difficult [$\mathbf{VK}_3$].

It turns out that the Lichnerowicz differential can be restricted to the complex associated with the normal vector bundle over each symplectic leaf of the Poisson manifold. The relationship between reduced cohomology and the Poisson cohomology of the whole manifold is the first interesting problem in this framework (it is not studied in the present paper). We are interested in another problem: how to calculate the reduced infinitesimal cohomology itself. Note that the infinitesimal Poisson cohomology appears in investigation of linearized Hamiltonian dynamics, deformation of Poisson bracket, and normal forms over a given symplectic leaf.

As shown in [**BV, Hu**], the Poisson calculus fits in the more general context of Lie algebroids. The reduction process and the calculation of the restricted cohomology have no specific features of the Poisson setting. The restricted Poisson calculus falls in the general scheme of transitive Lie algebroids, and the restricted Poisson cohomology is a cohomology with trivial coefficients of a certain transitive Lie algebroid.

The general procedure for calculating the cohomology of transitive Lie algebroids was developed by Mackenzie [**Mz**]. In the present paper we use a slightly modified approach and develop it for the case of trivial coefficients. For a special class of *quasiparallelizable* Lie algebroids we obtain simple formulas that are similar to formulas for the spectral sequence of a fiber bundle. We also pay special attention to Abelian algebroids because they correspond to the nondegenerate symplectic leaves and play a significant role in calculations related to non-Abelian algebroids. Since in some applications it is also significant to know the space of cocycles, we describe this space (of various dimensions) in the case of transitive Lie algebroids.

In the present paper we also treat the problem of equivalence of Poisson brackets having the same symplectic leaf. The first step is the classification up to formal equivalence, i.e., equivalence of Poisson tensors considered as formal power series in coordinates along the directions normal to the symplectic leaf. This problem has been investigated in detail only in the case when the symplectic leaf is a single point (see [$\mathbf{Co}_{1,2}$, **Du, Ly**]). We consider a formal equivalence problem for brackets having the same transitive Lie algebroid structure over the symplectic leaf. We give a sufficient condition for equivalence of such Poisson brackets in terms of cohomology of that transitive Lie algebroid with coefficients in a certain vector bundle.

The paper is organized as follows.

In §1 we give the necessary facts from Poisson calculus.

In §2 we give definitions for Lie algebroids and prove the restriction theorem.

In §3 we discuss some facts from the theory of connections in transitive Lie algebroids and introduce characteristic classes.

In §4 we calculate the cohomology of non-Abelian transitive Lie algebroids. For the quasiparallelizable Lie algebroids we present a simple formula for the second term of the spectral sequence in terms of the cohomology of a finite-dimensional normal algebra and the cohomology of a certain Abelian algebroid or the de Rham cohomology of the base. We show some applications of this formula and give sufficient conditions for the quasiparallelizability of a transitive Lie algebroid. We also describe the one- and two-dimensional spaces of cocycles.

In §5 we study the structure and the cohomology of Abelian algebroids. We present a classification theorem and calculate the space of cocycles and the cohomology space.

In §6 we discuss some properties of homogeneous Lie algebroids. By using the methods developed in §§ 4 and 5, we also calculate the cohomology for homogeneous algebroids over two different types of orbits of the coadjoint action of $E(3)$.

In §7 we consider a formal equivalence problem for Poisson brackets inducing the same transitive Lie algebroid structure over a symplectic leaf. We give a sufficient condition for formal equivalence in terms of cohomology classes of the transitive Lie algebroid with coefficients in a symmetric power of the normal algebra bundle. We also discuss calculation of these cohomology spaces.

§1. Coboundary Lichnerowicz operator over a symplectic leaf

All geometric objects considered in the present paper are infinitely differentiable, and we also assume that all linear spaces are over real numbers (although all the results remain valid over any field of zero characteristic).

Let $\mathcal{N}$ be a manifold equipped with the *Poisson bracket*

$$\{f, g\} = \Psi(df, dg), \qquad f, g, \in C^\infty(\mathcal{N}),$$

where Ψ is the corresponding antisymmetric tensor field, called sometimes a Poisson tensor [**MR**]. The tensor Ψ defines a morphism of vector bundles $q\colon T^*\mathcal{N} \to T\mathcal{N}$,

$$\langle \beta_2, q\beta_1 \rangle \stackrel{\text{def}}{=} \Psi(\beta_1, \beta_2), \qquad \beta_1, \beta_2 \in \Gamma(T^*\mathcal{N}). \tag{1.1}$$

(Here $\langle \cdot, \cdot \rangle$ is the pairing of 1-forms and vector fields.) It is well known [**Do, Ka, Ko**] that the Poisson bracket on functions can be extended to a bracket on 1-forms $\{\,,\,\}\colon \Gamma(T^*\mathcal{N}) \times \Gamma(T^*\mathcal{N}) \to \Gamma(T^*\mathcal{N})$,

$$\{\beta_1, \beta_2\} \stackrel{\text{def}}{=} \mathcal{L}_{q\beta_1}\beta_2 - \mathcal{L}_{q\beta_2}\beta_1 - d\Psi(\beta_1, \beta_2), \tag{1.2}$$

where $\mathcal{L}_{q\beta_1}\beta_2$ denotes the Lie derivative of the differential form β_2 along the vector field $q\beta_1$. The bracket (1.2) satisfies the Jacobi identity and endows the space of 1-forms with a Lie algebra structure; the mapping q is a homomorphism of this algebra into the Lie algebra of vector fields:

$$q\{\beta_1, \beta_2\} = [q\beta_1, q\beta_2]. \tag{1.3}$$

Moreover, we have the identities

$$d\{f, g\} = \{df, dg\}, \tag{1.4}$$

$$\{\beta_1, f\beta_2\} = f\{\beta_1, \beta_2\} + \big(\mathcal{L}_{q\beta_1}(f)\big)\beta_2, \qquad f, g, \in C^\infty(\mathcal{N}). \tag{1.5}$$

By $\mathcal{V}^k(\mathcal{N})$ we denote the space of contravariant antisymmetric tensor fields of the type $(k, 0)$ on the manifold $\mathcal{N}$. The *Schouten bracket* [**Sn**]

$$[\![\,,\,]\!]\colon \mathcal{V}^k(\mathcal{N}) \times \mathcal{V}^l(\mathcal{N}) \to \mathcal{V}^{k+l-1}(\mathcal{N})$$

defines the structure of a Lie superalgebra on $\mathcal{V}^\cdot(\mathcal{N}) \stackrel{\text{def}}{=} \bigoplus_{l=0}^{\dim \mathcal{N}} \mathcal{V}^k(\mathcal{N})$. It is well known [**Li**$_2$] that the Poisson tensor satisfies the condition $[\![\Psi,\Psi]\!] = 0$ and defines a *coboundary operator* $D\colon \mathcal{V}^k(\mathcal{N}) \to \mathcal{V}^{k+1}(\mathcal{N})$,

$$DQ \stackrel{\text{def}}{=} -[\![\Psi, Q]\!], \qquad Q \in \mathcal{V}^k(\mathcal{N}), \qquad D^2 = 0. \tag{1.6}$$

Note (see, for example, [**KSM**]) that the operator (1.6) is a standard coboundary operator [**F**] associated with the representation of the Lie algebra of differential forms in the ring $C^\infty(\mathcal{N})$:

$$DQ(\beta_0, \beta_1, \dots, \beta_k) = \sum_{j=0}^{k} (-1)^j \mathcal{L}_{q\beta_j}\big(Q(\beta_0, \beta_1, \dots, \widehat{\beta}_j, \dots, \beta_k)\big) + \sum_{0\le i<j\le k} (-1)^{i+j} Q\big(\{\beta_i, \beta_j\}, \beta_0, \beta_1, \dots, \widehat{\beta}_i, \dots, \widehat{\beta}_j, \dots, \beta_k\big).$$

(Here $Q \in \mathcal{V}^k(\mathcal{N})$, $\beta_j \in \Gamma(T^*\mathcal{N})$, and the "hat" stands over omitted terms.)

Let $\mathcal{O}$ be a *symplectic leaf* of the Poisson bracket (1.0), i.e., an integral leaf of the characteristic distribution $\operatorname{Ran}(q)$ [**We**]. The embedding $\mathcal{O} \hookrightarrow \mathcal{N}$ defines a vector bundle $T^*_{\mathcal{O}}\mathcal{N}$ over a symplectic leaf $\mathcal{O}$ with fiber $T^*_x\mathcal{N}$ ($x \in \mathcal{O}$). A restriction of differential forms to the leaf $\mathcal{O}$ is denoted by $r_{\mathcal{O}}\colon \Gamma(T^*\mathcal{N}) \to \Gamma(T^*_{\mathcal{O}}\mathcal{N})$. Similarly, we define the vector bundles $\bigwedge^k T_{\mathcal{O}}\mathcal{N}$ and the restriction operation $r^*_{\mathcal{O}}\colon \mathcal{V}^k(\mathcal{N}) \to \bigwedge^k T_{\mathcal{O}}\mathcal{N}$.

PROPOSITION 1.1. *The operation* $\{\,,\,\}_{\mathcal{O}}$ *is defined on sections of the bundle* $T^*_{\mathcal{O}}\mathcal{N}$, *and this operation gives a natural restriction of the bracket* (1.2),

$$r_{\mathcal{O}}\{\beta_1, \beta_2\} = \{r_{\mathcal{O}}\beta_1, r_{\mathcal{O}}\beta_2\}_{\mathcal{O}}.$$

COROLLARY 1.2. *The coboundary operator* $D_{\mathcal{O}}\colon \Gamma(\bigwedge^k T_{\mathcal{O}}\mathcal{N}) \to \Gamma(\bigwedge^{k+1} T_{\mathcal{O}}\mathcal{N})$ *that is a restriction of the Lichnerowicz operator* (1.6),

$$r^*_{\mathcal{O}} D = D_{\mathcal{O}} r^*_{\mathcal{O}}, \qquad D^2_{\mathcal{O}} = 0,$$

is defined on the sections of the bundle $\bigwedge^k T_{\mathcal{O}}\mathcal{N}$.

REMARK 1.3. This restriction can be performed over any Poisson submanifold.

The proof of Proposition 1.1 follows from the more general Theorem 2.1 in the next section.

§2. Reduction of Lie algebroids

A *Lie algebroid* [**Mz, Pr**] is a triple $(A \to B, q, \{\,,\,\})$, where $A \to B$ is a vector bundle, $q\colon A \to TB$ is a morphism of vector bundles, and $\{\,,\,\}\colon \Gamma(A)\times\Gamma(A) \to \Gamma(A)$ is a bilinear operation on sections that satisfies the following properties:

- the space of sections of the bundle A forms a Lie algebra under the operation $\{\,,\,\}$;
- the mapping $q\colon \Gamma(A) \to \mathcal{V}^1(B)$ is a homomorphism of Lie algebras:

$$q\{\alpha_1, \alpha_2\} = [q\alpha_1, q\alpha_2], \qquad \alpha_1, \alpha_2 \in \Gamma(A); \tag{2.1}$$

• any two sections $\alpha_1, \alpha_2 \in \Gamma(A)$ and each function $\varphi \in C^\infty(B)$ satisfy the relation

$$\{\alpha_1, \varphi\alpha_2\} = \varphi\{\alpha_1, \alpha_2\} + \big(\mathcal{L}_{q\alpha_1}(\varphi)\big)\alpha_2. \tag{2.2}$$

A Lie algebroid is called *transitive* if the morphism q is a surjection.

EXAMPLE 1. Obviously, the triple $(T^*\mathcal{N} \to \mathcal{N}, q, \{\ ,\ \})$, where $\mathcal{N}$ is a Poisson manifold, q is defined by (1.1), and $\{\ ,\ \}$ is defined by (1.2), is a Lie algebroid.

EXAMPLE 2 (Transformation algebroid [**Mz**]). Suppose that an action of a finite dimensional Lie algebra $(G, [\ ]_G)$ is given on the manifold B, that is, a homomorphism $q\colon G \to \mathcal{V}^1(B)$ is defined. The triplet $(G \times B \to B, q, \{\ ,\ \})$ is a Lie algebroid with commutator on sections defined as

$$\{\alpha_1, \alpha_2\} = [\alpha_1, \alpha_2]_G + \nabla^0_{q\alpha_1}\alpha_2 - \nabla^0_{q\alpha_2}\alpha_1. \tag{2.3}$$

(Here ∇^0 is a flat connection in the vector bundle $G \times B \to B$.)

An equivalent of the Poisson cohomology for a Lie algebroid is the cohomology of an infinite-dimensional Lie algebra $\Gamma(A)$ with coefficients in the $\Gamma(A)$-module $C^\infty(B)$ (i.e., the cohomology that corresponds to the representation $\alpha \mapsto \mathcal{L}_{q\alpha}$ in a one-dimensional trivial bundle).

Let us consider a standard complex [**F, Mz**] $C^k(A) \overset{\text{def}}{=} \Gamma(\wedge^k A^*)$, $k \geq 0$, and the differential operator $D\colon C^k(A) \to C^{k+1}(A)$,

$$\begin{aligned} Df(\alpha_0, \alpha_1, \dots, \alpha_k) &\overset{\text{def}}{=} \sum_{j=0}^{k}(-1)^j \mathcal{L}_{q\alpha_j}\big(f(\alpha_0, \alpha_1, \dots, \widehat{\alpha}_j, \dots, \alpha_k)\big) \\ &\quad + \sum_{0 \leq i < j \leq k}(-1)^{i+j} f\big(\{\alpha_i, \alpha_j\}, \alpha_0, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \widehat{\alpha}_j, \dots, \alpha_k\big). \end{aligned} \tag{2.4}$$

(Here $f \in C^k(A)$, $\alpha_j \in \Gamma(A)$, and the hat stands over omitted terms.) We denote the cocycles and the coboundaries of the operator (2.4) by $Z^k(A)$ and $B^k(A)$, respectively. The cohomology of the Lie algebroid [**Mz**] is defined as

$$\mathcal{H}^k(A) = Z^k(A)/B^k(A).$$

Note that on the cochains $C^k(A)$ the standard Grassmann multiplication is defined, with respect to which the operator (2.4) is antiderivation and the cohomology $\mathcal{H}^\cdot(A) \overset{\text{def}}{=} \bigoplus_k \mathcal{H}^k(A)$ inherits the structure of the Grassmann algebra [**F**]. The mapping $q^*\colon \Gamma(\bigwedge^k T^*B) \to C^k(A)$ is a homomorphism of Grassmann differential algebras (i.e., $q^*(\omega_1 \wedge \omega_2) = (q^*\omega_1) \wedge (q^*\omega_2)$ and $Dq^* = q^*d$), and thus, defines a homomorphism of Grassmann algebras $q^\dagger\colon H^\cdot(B) \to \mathcal{H}^\cdot(A)$.

For each submanifold $M \hookrightarrow B$, by $A_M \to M$ we denote the vector bundle obtained by the restriction of the bundle $A \to B$ to the base M; by $r_M\colon \Gamma(A) \to \Gamma(A_M)$ and $r_M^*\colon C^k(A) \to C^k(A_M)$ we denote the restrictions of sections of the corresponding bundles.

Let us consider a distribution $\mathcal{D}_x \overset{\text{def}}{=} (\operatorname{Ran} q_{|x}) \subset T_xB$. An *integral leaf* is a submanifold $\mathcal{O} \hookrightarrow B$ such that $T_y\mathcal{O} = \mathcal{D}_y$ for any point $y \in \mathcal{O}$. We shall say that an integral leaf $\mathcal{O}$ is *tame* if each section $\alpha \in \Gamma(A_\mathcal{O})$ of the restricted vector bundle has an extension $\widetilde{\alpha} \in \Gamma(A)$, $\alpha = r_\mathcal{O}\widetilde{\alpha}$.

§3. Connections, curvature, and characteristic classes in transitive Lie algebroids

Let $(A \to B, q, \{\,,\,\})$ be a transitive Lie algebroid. We consider a vector bundle $\mathfrak{g} \to B$, whose fiber is the kernel of the morphism q:

$$\mathfrak{g}_x \overset{\text{def}}{=} \operatorname{Ker} q_{|x} \tag{3.0}$$

(since the morphism q is surjective, $\mathfrak{g} = \bigcup_x \mathfrak{g}_x$ is a subbundle of the bundle $A \to B$). We have the following statement [**Mz**].

PROPOSITION 3.1. (a) *The space of sections* $\Gamma(\mathfrak{g})$ *is an ideal in the Lie algebra* $\Gamma(A)$.

(b) *The restriction of the bracket* $\{\,,\,\}$ *to the sections of the subbundle* $\mathfrak{g}$

$$[\theta_1, \theta_2] \overset{\text{def}}{=} \{\theta_1, \theta_2\}, \qquad \theta_1, \theta_2 \in \Gamma(\mathfrak{g}), \tag{3.1}$$

is linear over the ring of functions

$$[\theta_1, \varphi\theta_2] = \varphi[\theta_1, \theta_2], \qquad \varphi \in C^\infty(B), \tag{3.2}$$

and generates a Lie algebra structure in each fiber $\mathfrak{g}_x$; *the structures corresponding to different points of the base are isomorphic.*

PROOF. Statement (a) follows from formula (2.1). Formula (3.2) follows from (2.2). The algebras $\mathfrak{g}_x$ are isomorphic because there exists a Lie connection in the Lie algebroid (see below). □

COROLLARY 3.2. *The adjoint action of the Lie algebroid on the vector bundle* $\mathfrak{g}$

$$\operatorname{ad}_\beta \theta \overset{\text{def}}{=} \{\beta, \theta\}, \qquad \beta \in \Gamma(A), \quad \theta \in \Gamma(\mathfrak{g}), \tag{3.3}$$

has the following properties:

$$\operatorname{ad}_{\varphi\beta} \theta = \varphi \operatorname{ad}_\beta \theta, \qquad \varphi \in C^\infty(B), \tag{3.4}$$

$$\operatorname{ad}_\beta(\varphi\theta) = \varphi \operatorname{ad}_\beta \theta + \mathcal{L}_{q\beta}(\varphi)\theta, \tag{3.5}$$

$$\operatorname{ad}_\beta[\theta_1, \theta_2] = [\operatorname{ad}_\beta \theta_1, \theta_2] + [\theta_1, \operatorname{ad}_\beta \theta_2]. \tag{3.6}$$

Moreover, let us note that the subbundle $z\mathfrak{g}$ ($z\mathfrak{g}_x$ is the center of the Lie algebra $\mathfrak{g}_x$) and $[\mathfrak{g}, \mathfrak{g}]$ are invariant under the adjoint action (3.3).

A transitive Lie algebroid is called *Abelian*, if the Lie algebra $\mathfrak{g}_x$ is commutative. The algebroid $(T^*_{\mathcal{O}}\mathcal{N} \to \mathcal{O}, q, \{\,,\,\}_{\mathcal{O}})$ over a *nondegenerate* symplectic leaf $\mathcal{O}$ of the Poisson manifold $\mathcal{N}$ is an example of an Abelian algebroid (see [**We**]).

A linear connection ∇ in the vector bundle $\mathfrak{g} \to B$, satisfying the condition

$$\nabla_v[\theta_1, \theta_2] = [\nabla_v\theta_1, \theta_2] + [\theta_1, \nabla_v\theta_2], \qquad v \in \mathcal{V}^1(B). \tag{3.7}$$

is called a *Lie connection* [**Mz**] in a transitive Lie algebroid.

Clearly, this condition is equivalent to the fact that the parallel transport of the connection ∇, $T^\nabla_\gamma : \mathfrak{g}_{\gamma(0)} \to \mathfrak{g}_{\gamma(1)}$, along each path $\gamma\colon [0,1] \to B$ is an isomorphism of Lie algebras.

The *adjoint connection* [**Mz**] is the major example of the Lie connection. Let $\mathcal{P} \subset A$ be some subbundle transversal to the subbundle $\mathfrak{g}$:

$$A_x = \mathcal{P}_x \oplus \mathfrak{g}_x. \tag{3.8}$$

Obviously, each compact leaf is tame. The orbit of a coadjoint action in a "wild" Lie algebra [**Ki**$_1$] is an example of an integral leaf of the distribution $\operatorname{Ran} q$ that is not tame.

THEOREM 2.1. *Let the triple $(A \to B, q, \{\,,\,\})$ be a Lie algebroid and let $\mathcal{O}$ be a tame integral leaf of the distribution* $\operatorname{Ran} q$. *Then there is a natural structure of a transitive Lie algebroid $(A_{\mathcal{O}} \to \mathcal{O}, q, \{\,,\,\}_{\mathcal{O}})$ such that*

$$r_{\mathcal{O}}\{\alpha_1, \alpha_2\} = \{r_{\mathcal{O}}\alpha_1, r_{\mathcal{O}}\alpha_2\}_{\mathcal{O}}. \tag{2.5}$$

COROLLARY 2.2. *Let $D_{\mathcal{O}}$ be the coboundary operator* (2.4) *in the Lie algebroid $A_{\mathcal{O}}$. Then $r^*_{\mathcal{O}} D = D_{\mathcal{O}} r^*_{\mathcal{O}}$, and the homomorphism of Grassmann algebras $r^{\dagger}_{\mathcal{O}} \colon \mathcal{H}^{\cdot}(A) \to \mathcal{H}^{\cdot}(A_{\mathcal{O}})$ is well defined.*

REMARK 2.3. The theorem remains true for each tame submanifold $M \hookrightarrow B$ such that $D_x \subseteq T_x M$ for each $x \in M$.

To prove Theorem 2.1 we need the following observation.

LEMMA 2.4. *Suppose that a submanifold $M \hookrightarrow B$ satisfies the condition $\mathcal{D}_x \subseteq T_x M$ for each point $x \in M$. Then the linear subspace $J_M \stackrel{\text{def}}{=} \operatorname{Ker} r_M$ is an ideal in the Lie algebra $\Gamma(A)$. Moreover, the linear subspace $C_0^{\cdot}(A, M) \stackrel{\text{def}}{=} \operatorname{Ker} r^*_M$ is invariant under the action of the differential operator* (2.4).

PROOF. It suffices to prove this statement for a small open domain $U \subset B$ such that $U \cap M \neq \varnothing$. In the domain U there is a basis of sections $X_1, \dots, X_n \in \Gamma(A_U)$; the section $\alpha_0 = \sum_{i=1}^n a_i(x) X_i$ belongs to the space J_M if and only if $a_i(x)\big|_{x \in M} = 0$. For any section $\alpha \in \Gamma(A_U)$

$$r_M\{\alpha, \alpha_0\} = \sum_{i=1}^n \big(r_M a_i(x)\{\alpha, X_i\} + r_M \mathcal{L}_{q\alpha}(a_i) X_i\big) = 0$$

(since the vector field $q\alpha$ is tangent to the manifold M). Hence, $\{\alpha, \alpha_0\} \in J_M$. The second part of this lemma can be proved similarly. □

PROOF OF THEOREM 2.1. Since $\mathcal{O}$ is tame, for each section $\beta \in \Gamma(A_{\mathcal{O}})$ there exists its extension $\widetilde{\beta} \in \Gamma(A)$, $\beta = r_{\mathcal{O}}\widetilde{\beta}$. We define the operation $\{\alpha, \beta\}_{\mathcal{O}} \stackrel{\text{def}}{=} r_{\mathcal{O}}\{\widetilde{\alpha}, \widetilde{\beta}\}$, where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the extensions of the sections $\alpha, \beta \in \Gamma(A_{\mathcal{O}})$.

Now we show that this operation is well defined. Let $\widetilde{\beta}_1$, $\widetilde{\beta}_2$ and $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ be extensions of the sections β and α, respectively,

$$r_{\mathcal{O}}\{\widetilde{\alpha}_1, \widetilde{\beta}_1\} - r_{\mathcal{O}}\{\widetilde{\alpha}_2, \widetilde{\beta}_2\} = \frac{1}{2} r_{\mathcal{O}}\big(\{\widetilde{\alpha}_1 - \widetilde{\alpha}_2, \widetilde{\beta}_1 + \widetilde{\beta}_2\} + \{\widetilde{\alpha}_1 + \widetilde{\alpha}_2, \widetilde{\beta}_1 - \widetilde{\beta}_2\}\big) = 0$$

(since the sections $(\widetilde{\alpha}_1 - \widetilde{\alpha}_2)$ and $(\widetilde{\beta}_1 - \widetilde{\beta}_2)$ belong to the ideal $J_{\mathcal{O}} = \operatorname{Ker}(r_{\mathcal{O}})$). Obviously, the triple $(A_{\mathcal{O}} \to \mathcal{O}, q, \{\,,\,\}_{\mathcal{O}})$ is a transitive Lie algebroid. □

Since $\operatorname{Ker} q_{|x} = \mathfrak{g}_x$ and $\operatorname{Ran} q_{|x} = T_x B$, we have the isomorphism of vector bundles $p\colon TB \to \mathcal{P}$ that is inverse to the morphism q:

$$q \circ p = id_{\mathcal{V}^1(B)}. \tag{3.9}$$

The adjoint connection is defined via the adjoint action (3.3):

$$\nabla_v^{\mathcal{P}} \theta \stackrel{\text{def}}{=} \operatorname{ad}_{pv} \theta, \qquad v \in \mathcal{V}^1(B), \quad \theta \in \Gamma(\mathfrak{g}). \tag{3.10}$$

(Formulas (3.4), (3.5), and (3.9) imply that (3.10) actually defines a linear connection in the vector bundle $\mathfrak{g}$; that is, it satisfies (A.1) and (A.2).)

Now let us calculate the curvature (A.3) of the adjoint connection (3.10).

PROPOSITION 3.3. (a). *The curvature of the adjoint connection has the form*

$$K(v_1, v_2)\theta = [R^{\mathcal{P}}(v_1, v_2), \theta], \tag{3.11}$$

where the antisymmetric morphism of the vector bundles $R^{\mathcal{P}} : TB \otimes TB \to \mathfrak{g}$ is defined as

$$R^{\mathcal{P}}(v_1, v_2) \stackrel{\text{def}}{=} \{pv_1, pv_2\} - p[v_1, v_2], \qquad v_1, v_2 \in \mathcal{V}^1(B). \tag{3.11a}$$

(b) *The adjoint connection has zero curvature if and only if*

$$\{\Gamma(\mathcal{P}), \Gamma(\mathcal{P})\} \subseteq \Gamma(\mathcal{P} \oplus z\mathfrak{g}). \tag{3.12}$$

PROOF. We have

$$\begin{aligned} K(v_1, v_2)\theta &= \operatorname{ad}_{pv_1}(\operatorname{ad}_{pv_2} \theta) - \operatorname{ad}_{pv_2}(\operatorname{ad}_{pv_1} \theta) - \operatorname{ad}_{p[v_1,v_2]} \theta \\ &= \operatorname{ad}_{(\{pv_1,pv_2\} - p[v_1,v_2])} \theta = \operatorname{ad}_{R^{\mathcal{P}}(v_1,v_2)} \theta. \end{aligned}$$

By (2.1) and (3.9), $qR^{\mathcal{P}}(v_1, v_2) = 0$. Thus, statement (a) is proved.

To prove statement (b), we note that curvature (3.11) is identically equal to zero if and only if

$$R^{\mathcal{P}}(v_1, v_2) \in \Gamma(z\mathfrak{g}) \qquad \forall v_1, v_2 \in \mathcal{V}^1(B). \tag{3.13}$$

If condition (3.12) holds, then $\{pv_1, pv_2\} \in \Gamma(\mathcal{P} \oplus z\mathfrak{g})$ and (3.13) is satisfied because $\mathcal{P} \cap z\mathfrak{g} = \{0\}$. Conversely, the sections $\beta_1, \beta_2 \in \Gamma(\mathcal{P})$ can be represented as $\beta_j = pq\beta_j$; therefore,

$$\{\beta_1, \beta_2\} = \{pq\beta_1, pq\beta_2\} = \big(R^{\mathcal{P}}(q\beta_1, q\beta_2) + p[q\beta_1, q\beta_2]\big) \in \Gamma(\mathcal{P} \oplus z\mathfrak{g}). \quad \square$$

COROLLARY 3.4. *The restriction of the adjoint connection $\nabla^{\mathcal{P}}$ to the invariant subbundle $z\mathfrak{g}$ has zero curvature and is independent of the choice of a transversal subbundle $\mathcal{P}$. Thus it defines a universal flat connection in $z\mathfrak{g}$. In particular, in an Abelian algebroid the bundle $\mathfrak{g} \to B$ is flat.*

PROOF. By (3.11), the curvature is equal to zero. Now let us consider two subbundles $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ that are transversal to $\mathfrak{g}$, and the corresponding morphisms p and $\widetilde{p}$ (3.9). For each vector field $v \in \mathcal{V}^1(B)$ we have $(pv - \widetilde{p}v) \in \Gamma(\mathfrak{g})$; therefore, $\nabla_v^{\mathcal{P}}\theta - \nabla_v^{\widetilde{\mathcal{P}}}\theta = [pv - \widetilde{p}v, \theta] = 0$ for each section $\theta \in \Gamma(z\mathfrak{g})$. Hence, the adjoint connection in the subbundle $z\mathfrak{g}$ is independent of the choice of $\mathcal{P}$. $\square$

REMARK 3.5. In the book by Mackenzie [**Mz**] the morphism $p\colon TB \to \mathcal{P}$ from (3.9) is called a connection in the Lie algebroid. A curvature of this connection is $R^{\mathcal{P}}$, and this connection is called *flat* if $R^{\mathcal{P}}(v_1, v_2) \equiv 0$ (i.e., $\{\Gamma(\mathcal{P}), \Gamma(\mathcal{P})\} \subseteq \Gamma(\mathcal{P})$).

REMARK 3.6. In the context of Poisson geometry the adjoint connection appeared in [$\mathbf{VK}_3$].

Characteristic classes. To complete this section we would like to show that each transitive Lie algebroid structure over a base B gives a subring of $H^*(B)$ which is a generalization of characteristic classes for vector bundles.

Let $P^k(\mathfrak{g}_x) \stackrel{\text{def}}{=} S^k\mathfrak{g}_x^*$ denote the space of polynomials of degree k on the Lie algebra (3.0) (here S^k denotes the kth symmetric power). Consider a vector bundle $\operatorname{inv}^k(\mathfrak{g}) = \bigcup_{x\in B} \operatorname{inv}^k(\mathfrak{g}_x)$ whose fiber

(3.14)
$$\begin{aligned}\operatorname{inv}^k(\mathfrak{g}_x) \stackrel{\text{def}}{=} \{f \in P^k(\mathfrak{g}_x), (\ell_{\theta_0} f)(\theta_1, \theta_2, \dots, \theta_k) \\ = \sum_{j=1}^{k} f(\theta_1, \dots, \theta_{j-1}, [\theta_0, \theta_j], \theta_{j+1}, \dots, \theta_k) = 0,\ \theta_0, \theta_1, \dots, \theta_k \in \mathfrak{g}_x\}\end{aligned}$$

is a space of ad-invariant polynomials on $\mathfrak{g}_x$.

PROPOSITION 3.7. *Every adjoint connection* (3.10) *induces a flat connection in* $\operatorname{inv}^k(\mathfrak{g})$; *this connection does not depend on the choice of a transversal subbundle* $\mathcal{P}$.

PROOF. For each adjoint connection (3.10), a connection in a vector bundle $P^k(\mathfrak{g}) = \bigcup_{x\in B} P^k(\mathfrak{g}_x)$ is defined by the formula

$$(\nabla_u^{\mathcal{P}} s)(\theta_1, \dots, \theta_k) = \mathcal{L}_u\big(s(\theta_1, \dots, \theta_k)\big) - \sum_{j=1}^{k} s(\theta_1, \dots, \nabla_u^{\mathcal{P}}\theta_j, \dots \theta_k) \tag{3.15}$$

(here $s \in \Gamma(P^k(\mathfrak{g}))$ and $\theta_i \in \Gamma(\mathfrak{g})$). Now observe that for each $\theta \in \Gamma(\mathfrak{g})$ and $s \in \Gamma\big(P^k(\mathfrak{g})\big)$,

$$\ell_\theta \nabla_u^{\mathcal{P}} s = \nabla_u^{\mathcal{P}} \ell_\theta s - \ell_{\nabla_u \theta} s$$

(here ℓ_θ is defined as in (3.14); $u \in \mathcal{V}^1(B)$). Therefore, $\operatorname{inv}^k(\mathfrak{g})$ is an invariant subbundle in $P^k(\mathfrak{g})$ with respect to any adjoint connection (3.15), and thus we may regard the connection (3.15) as a connection on $\operatorname{inv}^k(\mathfrak{g})$. To see that this connection on $\operatorname{inv}^k(\mathfrak{g})$ has zero curvature, notice that

$$\big(K^{\nabla^{\mathcal{P}}}(u_1, u_2)\big)s = \ell_{R^{\mathcal{P}}(u_1,u_2)} s = 0 \qquad \text{for each } s \in \Gamma\big(\operatorname{inv}^k(\mathfrak{g})\big).$$

Now let $\mathcal{P}$ and $\mathcal{P}'$ be two different transversal subbundles (3.8), and let p and p' be the corresponding morphisms (3.9). Then

$$\nabla_u^{\mathcal{P}} s - \nabla_u^{\mathcal{P}'} s = \ell_{p'(u)-p(u)} s = 0 \qquad \forall\, s \in \Gamma(\operatorname{inv}^k(\mathfrak{g})), \quad u \in \mathcal{V}^1(B).$$

Therefore, the connection in $\operatorname{inv}^k(\mathfrak{g})$ does not depend on the choice of a transversal subbundle $\mathcal{P}$. □

Now consider a space $\operatorname{Inv}^k(\mathfrak{g}_x)$ of $\operatorname{Aut}(\mathfrak{g}_x)$-invariant polynomials on $\mathfrak{g}_x$:

$$\begin{aligned}\operatorname{Inv}^k(\mathfrak{g}_x) \stackrel{\text{def}}{=} \{f \in P^k(\mathfrak{g}_x),\ f(g\cdot\theta_1, g\cdot\theta_2, \dots, g\cdot\theta_k) = f(\theta_1, \theta_2 \dots, \theta_k), \\ \theta_1, \theta_2, \dots, \theta_k \in \mathfrak{g}_x, \forall\, g \in \operatorname{Aut}(\mathfrak{g}_x)\}\end{aligned} \tag{3.16}$$

(here $\mathrm{Aut}(\mathfrak{g}_x)$ is the group of all automorphisms of the Lie algebra $\mathfrak{g}_x$, and $g \cdot \theta$ denotes the action of $g \in \mathrm{Aut}(\mathfrak{g}_x)$ on $\theta \in \mathfrak{g}_x$).

It is easy to see that $\mathrm{Inv}^k(\mathfrak{g}_x)$ is naturally imbedded into the space of parallel sections of the vector bundle $\mathrm{inv}^k(\mathfrak{g})$; therefore, for each $f \in \mathrm{Inv}^k(\mathfrak{g}_x)$ we can define a differential form $c_f \in \Gamma(\Lambda^{2k}T^*B)$,

(3.17)
$$c_f(u_1, u_2, \dots, u_{2k}) = f(R^{\mathcal{P}}, \dots, R^{\mathcal{P}}) \stackrel{\text{def}}{=} \sum_{\sigma \in S_{2k}} (-1)^\sigma \frac{1}{(2k)!} f\big(R^{\mathcal{P}}(u_{\sigma(1)}, u_{\sigma(2)}), \dots, R^{\mathcal{P}}(u_{\sigma(2k-1)}, u_{\sigma(2k)})\big),$$

(here the sum is taken over all permutations of $\{1, \dots, 2k\}$, $u_1, \dots, u_{2k} \in \mathcal{V}^1(B)$, and $R^{\mathcal{P}}$ is a "curvature" (3.11a) of some adjoint connection).

THEOREM 3.8. *For each $f \in \mathrm{Inv}^k(\mathfrak{g}_x)$, a differetial form c_f (3.17) is closed and its cohomology class in $H^{2k}(B)$ does not depend on the choice of a transversal subbundle $\mathcal{P}$.*

PROOF. Since each $f \in \mathrm{Inv}^k(\mathfrak{g}_x)$ is a parallel section of $\mathrm{inv}^k(\mathfrak{g})$ and $\nabla^{\mathcal{P}} R^{\mathcal{P}} = 0$ (see [**Mz**]), we have

$$dc_f = df(R^{\mathcal{P}}, R^{\mathcal{P}}, \dots, R^{\mathcal{P}}) = \sum_{j=1}^{k} f(R^{\mathcal{P}}, \dots, \nabla R^{\mathcal{P}}, \dots, R^{\mathcal{P}}) + (\nabla^{\mathcal{P}} f)(R^{\mathcal{P}}, \dots, R^{\mathcal{P}}) = 0.$$

Now we show that the cohomology class of c_f does not depend on the choice of a transversal subbundle $\mathcal{P}$. Let $\mathcal{P}$ and $\mathcal{P}'$ be distinct subbundles in A transversal to $\mathfrak{g}$ (3.8), and let p and p' be the corresponding morphisms (3.9). Define

$$p^t(u) \stackrel{\text{def}}{=} (1-t)p(u) + tp'(u).$$

It is easy to see that for each $t \in \mathbb{R}$, $p^t(u)$ satisfies (3.9) and thus defines a transversal subbundle $\mathcal{P}^t$ for each $t \in \mathbb{R}$.

Let R^t denote the curvature (3.11a) of the transversal subbundle $\mathcal{P}^t$, and for each $f \in \mathrm{Inv}^k(\mathfrak{g}_x)$ define $c_f^t = f(R^{\mathcal{P}^t}, \dots, R^{\mathcal{P}^t})$, where the right-hand side is defined as the right-hand side of (3.17) with $R^{\mathcal{P}}$ replaced by $R^{\mathcal{P}^t}$. This differential form is closed for each $t \in \mathbb{R}$.

In order to show that the cohomology class of c_f^t does not depend on t, consider another transitive algebroid $(A_1 \to B \times \mathbb{R}, q_1, \{\,\}_1)$, where $A_1 = \pi_1^* A \oplus \pi_2^* T\mathbb{R}$ (here π_1 and π_2 are natural projections of $B \times \mathbb{R}$ onto B and $\mathbb{R}$ respectively),

$$q_1(\alpha \oplus v) = q(\alpha) + v, \qquad \alpha \in \Gamma(A), \quad v \in \mathcal{V}^1(\mathbb{R}),$$
$$\{\alpha_1 \oplus v_1, \alpha_2 \oplus v_2,\}_1 = \{\alpha_1, \alpha_2\} \oplus [v_1, v_2], \quad \forall \alpha_i \in \Gamma(A), \quad v_i \in \mathcal{V}^1(\mathbb{R}), \quad i = 1, 2.$$

Introduce a vector bundle morphism $p_1 \colon T(B \times \mathbb{R}) \to A_1$,

$$p_1(u + v) = \big((1-t)p(u) + tp'(u)\big) \oplus v,$$

where t is the coordinate on $\mathbb{R}$, $u \in \mathcal{V}^1(B)$, and $v \in \mathcal{V}^1(\mathbb{R})$. It is easy to see that $q_1 \circ p_1 = \mathrm{id}_{T(B \times \mathbb{R})}$; thus p_1 defines a subbundle in A_1 transversal to $\mathfrak{g}_1 = \ker q_1 =$

$\pi_1^*\mathfrak{g}$. Direct calculation shows that the curvature $R^{\mathcal{P}_1}$ (3.11a) of this transversal subbundle satisfies

$$R^{\mathcal{P}_1}(u_1+v_1,u_2+v_2)=R^{\mathcal{P}'}(u_1,u_2)+\mathcal{L}_{v_1}(t)(p'(u_2)-p(u_2)) - \mathcal{L}_{v_2}(t)(p'(u_1)-p(u_1)) \tag{3.18}$$

(here $u_i \in \mathcal{V}^1(B)$ and $v_i \in \mathcal{V}^1(\mathbb{R})$, $i=1,2$).

Since $\mathfrak{g}_1 = \ker q_1 = \pi_1^*\mathfrak{g}$, it follows that $\operatorname{Inv}^k(\mathfrak{g}_{1x}) = \operatorname{Inv}^k(\mathfrak{g}_x)$; therefore for each $f \in \operatorname{Inv}^k(\mathfrak{g}_x)$ we can define $\omega_f = f(R_1,\dots,R_1)$ using formula (3.17). This is a closed differential form on $B\times\mathbb{R}$, and using (3.18) it is easy to see that $c_f^t = i_t^*\omega_f$, where $i_t\colon B\to B\times\mathbb{R}$ is defined by the formula

$$i_t(x)=(x,t),\qquad x\in B,\quad t\in\mathbb{R}.$$

Since all the i_t are homotopic, the cohomology classes of $i_t^*\omega_f$ coincide for all $t\in\mathbb{R}$; in particular, the cohomology classes of $c_f^0 = f(R^{\mathcal{P}},\dots,R^{\mathcal{P}})$ and $c_f^1 = f(R^{\mathcal{P}'},\dots,R^{\mathcal{P}'})$ coincide. This proves that the cohomology class of (3.17) does not depend on the choice of a transversal subbundle. □

EXAMPLE (Chern characteristic classes of a vector bundle). Let $\mathcal{E}\to B$ be a vector bundle. Consider the transitive algebroid $A=\operatorname{CDO}(\mathcal{E})$ of covariant differential operators [**Mz**]. Every section α of A acts on $\Gamma(\mathcal{E})$ as a first-order differential operator $\alpha(fs)=\mathcal{L}_{q(\alpha)}(f)s+f\alpha(s)$. It is easy to see that each morphism $p\colon TB\to\operatorname{CDO}(\mathcal{E})$ satisfying (3.9) defines a linear connection ∇^p on $\mathcal{E}$ ($\nabla_u^p s = p(u)(s)$) and the curvature (3.11a) corresponding to p is exactly the curvature (A.3) of the connection ∇^p. This proves that the cohomology classes c_f are exactly characteristic classes of the vector bundle $\mathcal{E}$ (see, for example, [**NS**], Appendix C).

§4. Calculation of cohomology of a transitive Lie algebroid

Let $(A\to B, q, \{\ ,\ \})$ be a transitive Lie algebroid. We will always assume that the bundle $\mathfrak{g}$ (3.0) can be decomposed into the direct sum of subbundles $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{h}$, which satisfies two conditions:

- $\mathfrak{g}_0$ and $\mathfrak{h}$ are invariant under the adjoint action (3.3);
- $\mathfrak{g}_0$ is a subbundle of the center $\mathfrak{g}_0\subset z\mathfrak{g}$.

In what follows, we are interested in two extreme cases: (1) $\mathfrak{h}=\mathfrak{g}$ and (2) $\mathfrak{g}_0 = z\mathfrak{g}$ (though, generally speaking, not every transitive Lie algebroid has an ad-invariant subbundle $\mathfrak{h}$ that is transversal to $z\mathfrak{g}$).

Let us choose some transversal subbundle $\mathcal{P}$ (3.8). We consider a vector bundle

$$A_0 \overset{\text{def}}{=} \mathcal{P}\oplus\mathfrak{g}_0 \simeq TB\oplus\mathfrak{g}_0, \tag{4.0}$$

on which we define the structure of an Abelian algebroid $(A_0\to B, q, \{\ ,\ \}_0)$. A bracket on sections of the subbundle $A_0\subset A$ is defined by the formula

$$\{\alpha_1,\alpha_2\}_0 \overset{\text{def}}{=} \{\alpha_1,\alpha_2\} - \pi\{\alpha_1,\alpha_2\}, \tag{4.1}$$

where $\pi\colon A\to\mathfrak{h}$ is the projection on the subbundle $\mathfrak{h}$ along the subbundle A_0.

By $\mathcal{K}^{s,t}$ we denote the space of s-cochains of the Lie algebroid A_0 with values in the space of sections of the t-th external power of the subbundle dual to $\mathfrak{h}$:

$$\mathcal{K}^{s,t} \overset{\text{def}}{=} \Gamma\Big(\bigwedge^t \mathfrak{h}^*\otimes\bigwedge^s A_0^*\Big). \tag{4.2}$$

The space $\mathcal{K}^\cdot = \bigoplus_k \mathcal{K}^k$, where $\mathcal{K}^k = \bigoplus_{s+t=k} \mathcal{K}^{s,t}$ is a Grassmann algebra with respect to the multiplication

$$(\eta_1 \otimes \omega_1) \square (\eta_2 \otimes \omega_2) = (-1)^{s_1+t_2}(\eta_1 \wedge \eta_2) \otimes (\omega_1 \wedge \omega_2), \tag{4.3}$$

where $\eta_i \in \Gamma(\wedge^{t_i}\mathfrak{h}^*)$, $\omega_i \in C^{s_i}(A_0)$, $i = 1, 2$.

We define the mappings $p^{s,t}\colon C^{t+s}(A) \to \mathcal{K}^{s,t}$ and $e^{s,t}\colon \mathcal{K}^{s,t} \to C^{t+s}(A)$ as follows:

$$p^{s,t} f(\theta_1, \dots, \theta_t)(\beta_1, \dots, \beta_s) \stackrel{\text{def}}{=} f(\theta_1, \dots, \theta_t, \beta_1, \dots, \beta_s) \tag{4.4}$$

(here $f \in C^{t+s}(A)$, $\theta_j \in \Gamma(\mathfrak{h})$, and $\beta_l \in \Gamma(A_0)$) and

$$e^{s,t}\eta(\alpha_1, \dots, \alpha_{t+s}) \stackrel{\text{def}}{=} \frac{1}{t!s!} \sum_{\sigma \in S_{t+s}} (-1)^\sigma \eta\big(\pi\alpha_{\sigma(1)}, \dots, \pi\alpha_{\sigma(t)}\big)\big(\alpha_{\sigma(t+1)} - \pi\alpha_{\sigma(t+1)}, \dots, \alpha_{\sigma(t+s)} - \pi\alpha_{\sigma(t+s)}\big)$$

(here $\eta \in \mathcal{K}^{s,t}$ and $\alpha_l \in \Gamma(A)$, $l = 1, \dots, t+s$, and the summation is taken over the set of permutations).

We write $p^k \stackrel{\text{def}}{=} \sum_{t+s=k} p^{s,t}$ and $e^k \stackrel{\text{def}}{=} \sum_{t+s=k} e^{s,t}$.

Proposition 4.1. *The mapping $p^k : C^k(A) \to \mathcal{K}^k$ is an isomorphism of Grassmann algebras*

$$(p^k)^{-1} = e^k, \quad p^{k_1+k_2}(f_1 \wedge f_2) = p^{k_1} f_1 \square p^{k_2} f_2 \quad \big(f_i \in C^{k_i}(A),\ i = 1, 2\big), \tag{4.5}$$

satisfying the commutation relation

$$p^{k+1} D = (d_0 + d_1 + \delta)p^k, \tag{4.6}$$

where the operators $d_0\colon \mathcal{K}^{s,t} \to \mathcal{K}^{s,t+1}$, $d_1\colon \mathcal{K}^{s,t} \to \mathcal{K}^{s+1,t}$, and $\delta\colon \mathcal{K}^{s,t} \to \mathcal{K}^{s+2,t-1}$ are defined as follows:

(4.7)
$$d_0\eta(\theta_0, \theta_1, \dots, \theta_t)(\alpha_1, \dots, \alpha_s) \stackrel{\text{def}}{=} \sum_{0 \le i < j \le t} (-1)^{i+j} \eta\big([\theta_i, \theta_j], \theta_0, \theta_1, \dots, \widehat{\theta}_i, \dots, \widehat{\theta}_j, \dots, \theta_t\big)(\alpha_1, \dots, \alpha_s),$$

$$d_1 \stackrel{\text{def}}{=} \sum_{t,s} (-1)^t \overline{\nabla}^{s,t}, \tag{4.8}$$

(4.9)
$$\begin{aligned}\overline{\nabla}^{s,t}\eta(\theta_1, \dots, \theta_t)(\alpha_0, \alpha_1, \dots, \alpha_s) &\stackrel{\text{def}}{=} \sum_{j=0}^{s} (-1)^j \bigg(\mathcal{L}_{q\alpha_j}\big(\eta(\theta_1, \dots, \theta_t)(\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_s)\big) \\ &- \sum_{\tau=1}^{t} \eta(\theta_1, \dots, \theta_{\tau-1}, \nabla^{\mathcal{P}}_{q\alpha_j}\theta_\tau, \theta_{\tau+1} \dots, \theta_t)(\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_s)\bigg) \\ &+ \sum_{0 \le l < m \le s} (-1)^{l+m} \eta(\theta_1, \dots, \theta_t)(\{\alpha_l, \alpha_m\}_0, \alpha_0, \alpha_1, \dots, \widehat{\alpha}_l, \dots, \widehat{\alpha}_m, \dots, \alpha_s),\end{aligned}$$

(4.10)

$$\delta\eta(\theta_1,\dots,\theta_{t-1})(\alpha_0,\alpha_1,\dots,\alpha_s,\alpha_{s+1}) \overset{\text{def}}{=} \sum_{0\le l<m\le s+1} (-1)^{l+m} \times \eta\big(\pi R^{\mathcal{P}}(q\alpha_l,q\alpha_m),\theta_1,\dots,\theta_{t-1}\big)(\alpha_0,\alpha_1,\dots,\widehat{\alpha}_l,\dots,\widehat{\alpha}_m,\dots,\alpha_s,\alpha_{s+1}),$$

(here π is the projection on the subbundle $\mathfrak{h}$ along $\mathfrak{g}_0$) and satisfy the following commutation relations:

$$d_0^2 = 0, \tag{4.11}$$

$$d_0d_1 + d_1d_0 = 0, \tag{4.12}$$

$$d_1^2 + d_0\delta + \delta d_0 = 0, \tag{4.13}$$

$$d_1\delta + \delta d_1 = 0, \tag{4.14}$$

$$\delta^2 = 0. \tag{4.15}$$

PROOF. Formulas (4.5)–(4.10) can be verified by direct calculations. Each formula (4.1i), $i = 1,2,\dots,5$, follows from the identity $p^{s+i-1,t+3-i}D^2e^{s,t} = 0$. □

REMARK 4.2. The operator d_0 is a standard coboundary operator for the zero representation of the Lie algebra $\Gamma(\mathfrak{h})$ in the space $C^{\cdot}(A_0) = \bigoplus C^s(A_0)$. The operator d_1 coincides up to the sign with the differential operator (A.4) (see Appendix) that corresponds to the connection in the bundle $\bigoplus_t \bigwedge^t \mathfrak{h}^*$ induced by the adjoint connection $\nabla^{\mathcal{P}}$. In particular, for any section $\eta \in \mathcal{K}^{s,0}$ we have $d_1\eta = D_0\eta$, where D_0 is the coboundary operator (2.4) of the Abelian algebroid $(A_0 \to B, q, \{\ ,\ \}_0)$.

REMARK 4.3. Each of the operators d_0, d_1, and δ is antidifferential with respect to Grassmann multiplication (4.3):

$$(d_0+d_1+\delta)(\eta_1 \,\square\, \eta_2) = \big((d_0+d_1+\delta)\eta_1\big) \,\square\, \eta_2 + (-1)^{s_1+t_1}\eta_1 \,\square\, (d_0+d_1+\delta)\eta_2,$$

where $\eta_i \in \mathcal{K}^{s_i,t_i}$, $i = 1,2$.

Let us consider the Hochschild–Serre spectral sequence $(E_r^{s,t}, d_r^{s,t})$ [**HS**], related to the ideal $\Gamma(\mathfrak{h})$ of the Lie algebra $\Gamma(A)$. We have the following result (see [**Mz**]).

PROPOSITION 4.4. *The spectral sequence $(E_r^{s,t}, d_r^{s,t})$ converges to the cohomology space $\mathcal{H}^{\cdot}(A)$:*

$$\mathcal{H}^k(A) \simeq \bigoplus_{t+s=k} E_r^{s,t} \qquad \textit{for} \quad r \ge \max(t+1,s)+1, \tag{4.16}$$

and

$$(E_0^{s,t}, d_0^{s,t}) \simeq (\mathcal{K}^{s,t}, d_0),$$

$$(E_1^{s,t}, d_1^{s,t}) \simeq (H_{d_0}\mathcal{K}^{s,t}, d_1), \tag{4.17}$$

$$(E_2^{s,t}, d_2^{s,t}) \simeq (H_{d_1}H_{d_0}\mathcal{K}^{s,t}, \delta - d_1 d_0^{-1} d_1), \tag{4.18}$$

where the operators d_0, d_1, and δ are defined by (4.7)–(4.10).

The proof is standard [**HS**].

Let us consider a vector bundle $H^t\mathfrak{h} \to B$, whose fiber is the cohomololgy space $H^t(\mathfrak{h}_x)$ of the Lie algebra $\mathfrak{h}_x$. It is easy to see that the space of sections of this bundle is isomorphic to the space $H_{d_0}\mathcal{K}^{0,t}$ (the isomorphism can be established by some scalar product in the bundle $\bigwedge^t \mathfrak{h}^*$; in this case the fiber $H^t\mathfrak{h}_{|x}$ is realized as the orthogonal complement to the subbundle of coboundaries in the subbundle of cocycles). Similarly, one can obtain the isomorphism

$$E_1^{s,t} \simeq H_{d_0}\mathcal{K}^{s,t} \simeq \Gamma(H^t\mathfrak{h} \otimes \bigwedge^s A_0^*). \tag{4.19}$$

By ∇^t we denote the connection in the vector bundle $\bigwedge^t \mathfrak{h}^*$ induced by the adjoint connection. If we restrict formula (4.12) to the space $\mathcal{K}^{0,t}$, then we see that the connection ∇^t satisfies the formula

$$\nabla_v^{t+1} d_0\eta = d_0\nabla_v^t\eta \qquad \forall\, \eta \in \mathcal{K}^{0,t}, \quad v \in \mathcal{V}^1(B).$$

Hence, the connection ∇^t is invariant over the subbundle of cocycles and the subbundle of coboundaries of the operator d_0. Thus this connection induces a connection in the vector bundle $H^t\mathfrak{h}$, which we denote by the same symbol ∇^t.

The following statement is a reformulation of the result obtained in [**Mz**].

PROPOSITION 4.5. (a) *In the bundle $H^t\mathfrak{h}$ the connection ∇^t, which is induced by the adjoint connection $\nabla^{\mathcal{P}}$, is flat and independent of the choice of the transversal subbundle $\mathcal{P}$.*

(b) *The second term of the spectral sequence coincides with the cohomology space of the operator $\overline{\nabla}^t$ defined as in* (A.6) *associated with the flat connection ∇^t:*

$$E_2^{s,t} \simeq H^s_{\nabla^t}(A_0, H^t\mathfrak{h}). \tag{4.20}$$

PROOF. We define a morphism of vector bundles $\ell\colon \mathfrak{h} \otimes \bigwedge^t \mathfrak{h}^* \to \bigwedge^t \mathfrak{h}^*$ by the formula

$$\ell_\theta\eta(\theta_1, \dots, \theta_t) \overset{\text{def}}{=} \sum_{\tau=1}^{t} \eta(\theta_1, \dots, \theta_{\tau-1}, [\theta, \theta_\tau], \theta_{\tau+1}, \dots, \theta_t).$$

For each cohomology class $[\eta]_{d_0} \in H_{d_0}\mathcal{K}^{0,t}$ we have $[\ell_\theta\eta]_{d_0} = 0$ (see, for example, [**CE**]). One can easily verify that

$$(\nabla_{v_1}^t\nabla_{v_2}^t - \nabla_{v_2}^t\nabla_{v_1}^t - \nabla_{[v_1,v_2]}^t)\eta = -\ell_{\pi R^{\mathcal{P}}(v_1,v_2)}(\eta) \qquad \forall\, \eta \in \mathcal{K}^{0,t}.$$

Therefore, in the bundle $H^t\mathfrak{h}$ the curvature of the connection ∇^t is equal to zero.

Let $\mathcal{P}$ and $\mathcal{P}'$ be two subbundles (3.8) transversal to $\mathfrak{g}$, let p and p' be morphisms (3.9), and let ∇^t and $\nabla^{t'}$ be the corresponding connections in the bundle $\bigwedge^t \mathfrak{h}^*$. Obviously, $(pv - p'v) \in \Gamma(\mathfrak{g})$ for any vector field $v \in \mathcal{V}^1(B)$. It is easy to see that

$$(\nabla_v^t - \nabla_v^{t'})\eta = \ell_{\pi(pv-p'v)}\eta \qquad \forall\, \eta \in \Gamma(\bigwedge^t \mathfrak{h}^*).$$

Therefore, the adjoint connections $\nabla^{\mathcal{P}}$ and $\nabla^{\mathcal{P}'}$ induce the same connection in the bundle $H^t\mathfrak{h}$. Statement (a) is thereby proved.

Statement (b) follows from (4.17), (4.19), (4.9), and statement (a). □

A simple consequence of formula (4.20) is the following generalization of the theorem stating that the de Rham cohomology is finite-dimensional.

PROPOSITION 4.6. *Suppose the base B of a transitive Lie algebroid is a finite-type manifold. Then the cohomology space $\mathcal{H}^{\cdot}(A)$ is finite-dimensional.*

PROOF. Let us consider the spectral sequence corresponding to the invariant subbundle $\mathfrak{h} = \mathfrak{g}$. Then we have $E_2^{s,t} \simeq H^s_{\nabla^t}(H^t\mathfrak{g})$ (see the definition for the cohomology of the flat connection in the Appendix). By applying Lemma A.3, we prove that the second term of this spectral sequence is finite-dimensional. Hence, the space $\mathcal{H}^k(A) \simeq \bigoplus_{s+t=k} E_\infty^{s,t}$ is also finite-dimensional. □

A transitive Lie algebroid is called *quasiparallelizable* if there exists a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{h}$ into the direct sum of ad-invariant subbundles such that $\mathfrak{g}_0 \subset z\mathfrak{g}$, and for any $t \geq 1$ the canonical connection ∇^t has a trivial holonomy group in the bundle $H^t\mathfrak{h}$.

Obviously, each transitive Lie algebroid over a simply connected base B is quasiparallelizable ($\mathfrak{h} = \mathfrak{g}$). It also should be mentioned that we can ensure the quasiparallelizability of transitive Lie algebroids by imposing some conditions on the Lie algebra $\mathfrak{g}_x$.

PROPOSITION 4.7. *For a Lie algebroid to be quasiparallelizable, it suffices that the following two conditions hold simultaneously.*

(i) *The Lie algebra $\mathfrak{g}_x$ can be decomposed into the direct sum of its center and its commutant, $\mathfrak{g}_x = z\mathfrak{g}_x \oplus [\mathfrak{g}_x, \mathfrak{g}_x]$.*

(ii) *The Lie algebra $\mathfrak{h}_x = [\mathfrak{g}_x, \mathfrak{g}_x]$ is complete (that is, each derivation of the Lie algebra $\mathfrak{h}_x$ is interior) and the group* $\operatorname{Aut}(\mathfrak{h}_x)$ *of automorphisms of the Lie algebra $\mathfrak{h}_x$ is connected.*

PROOF. For each path $\gamma\colon [0,1] \to B$, $\gamma(0) = \gamma(1) = x_0$, the parallel transport $T_\gamma\colon H^t(\mathfrak{h}_{x_0}) \to H^t(\mathfrak{h}_{x_0})$ has the form $T_\gamma[C]_{d_0} = [T^t_\gamma C]_{d_0}$, where the parallel transport $T^t_\gamma\colon C^t(\mathfrak{h}_{x_0}) \to C^t(\mathfrak{h}_{x_0})$ satisfies the formula

$$(T^t_\gamma C)(\theta_1, \dots, \theta_t) = C(T^{\mathcal{P}}_\gamma \theta_1, \dots, T^{\mathcal{P}}_\gamma \theta_t), \qquad \theta_1, \dots, \theta_t \in \mathfrak{h}_{x_0},$$

and $T^{\mathcal{P}}_\gamma\colon \mathfrak{h}_{x_0} \to \mathfrak{h}_{x_0}$ is the parallel transport of the adjoint connection $\nabla^{\mathcal{P}}$ in the invariant subbundle $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$.

Obviously, $T^{\mathcal{P}}_\gamma$ is an automorphism of the Lie algebra $\mathfrak{h}_x$. It follows from condition (ii) that each automorphism $a \in \operatorname{Aut}(\mathfrak{h}_x)$ can be represented in the form of a finite product of automorphisms of the form $a = \exp(\operatorname{ad}\theta)$, where $\theta \in \mathfrak{h}_x$ (see, e.g., Proposition 4.6.1 in **[GG]**). Hence, to complete the proof, it remains to show that the group of interior automorphisms acts *trivially* on the space $H^t(\mathfrak{h}_{x_0})$.

We consider the one-parametric group of linear isomorphisms $a_\tau\colon H^t(\mathfrak{h}_{x_0}) \to H^t(\mathfrak{h}_{x_0})$, $a_\tau[C]_{d_0} \stackrel{\text{def}}{=} [C \cdot \operatorname{Ad}(\exp(\tau\theta))]_{d_0}$, where $\theta \in \mathfrak{h}_{x_0}$ and

$$(C \cdot \operatorname{Ad}(\exp(\tau\theta)))(\theta_1, \dots, \theta_t) \stackrel{\text{def}}{=} C(e^{\operatorname{ad}\tau\theta}\theta_1, \dots, e^{\operatorname{ad}\tau\theta}\theta_t).$$

As is easy to see,

$$\begin{aligned}
&\frac{d}{d\tau}\big(C \cdot \operatorname{Ad}(\exp(\tau\theta))\big)(\theta_1, \dots, \theta_t) \\
&\quad = \sum_{l=1}^{t} C\big(e^{\operatorname{ad}\tau\theta}\theta_1, \dots, e^{\operatorname{ad}\tau\theta}\theta_{l-1}, [\theta, e^{\operatorname{ad}\tau\theta}\theta_l], e^{\operatorname{ad}\tau\theta}\theta_{l+1}, \dots, e^{\operatorname{ad}\tau\theta}\theta_t\big) \\
&\quad = \ell_\theta\big(C \cdot \operatorname{Ad}(\exp(\tau\theta))\big)(\theta_1, \dots, \theta_t).
\end{aligned}$$

Since $[\ell_\theta C]_{d_0} = 0$ for each cocycle $C \in Z_{d_0}(\mathfrak{h}_x)$ (see [**CE**]), we have $\frac{d}{d\tau} a_\tau = 0$; hence, $a_\tau \equiv a_0 = \mathrm{id}$. The proposition is proved. □

THEOREM 4.8. *Let $(A \to B, q, \{\ ,\ \})$ be a quasiparallelizable Lie algebroid. Then the second term of the spectral sequence, corresponding to the subbundle $\mathfrak{h}$, has the form*

$$E_2^{s,t} \simeq \mathbf{h}^t \otimes \mathcal{H}^s(A_0), \tag{4.21}$$

where $\mathcal{H}^s(A_0)$ is the cohomology space of the Abelian algebroid $(\mathcal{P} \oplus \mathfrak{g}_0 \to B, q, \{\ ,\ \}_0)$ and $\mathbf{h}^t \cong H^t \mathfrak{h}_{|x}$ is the cohomology space of a finite-dimensional Lie algebra $\mathfrak{h}_x$. In this case for any $\zeta \in \mathbf{h}^t$, $\varkappa \in \mathcal{H}^s(A_0)$, we have

$$d_2^{s,t}(\zeta \otimes \varkappa) = d_2^{0,t}\zeta \boxtimes \varkappa, \tag{4.22}$$

where the operation of Grassmann multiplication $\boxtimes$ in the space $E_2^{s,t}$ is induced by the Grassmann multiplication (4.3) *in the space $E_0^{s,t}$.*

PROOF. By applying Lemma A.4. (see the Appendix) to (4.20), we obtain (4.21). Formula (4.22) follows from (4.18) and Remark 4.3. □

COROLLARY 4.9. *Let $d_k^{s,t}[\zeta \otimes \varkappa]_k = 0$ for any $k = 2, 3, \ldots, r-1$. Then*

$$d_r^{s,t}[\zeta \otimes \varkappa]_r = (d_r^{0,t}[\zeta]_r) \boxtimes [\varkappa]_r,$$

where $[\cdot]_r$ is the cohomology class in the space $E_r^{s,t} = \operatorname{Ker} d_{r-1}^{s,t} / \operatorname{Ran} d_{r-1}^{s-r,t+r-1}$.

REMARK 4.10. If $\mathfrak{h} = \mathfrak{g}$, i.e., $A_0 = \mathcal{P} \simeq TB$, then the mapping $q^\dagger : \mathcal{H}^k(A_0) \to H^k(B)$ is an isomorphism of Grassmann algebras.

COROLLARY 4.11. *Let the base B of a transitive Lie algebroid be a simply connected finite-type manifold. Then the Euler characteristic*

$$\chi(A) \overset{\mathrm{def}}{=} \sum_{k=0}^{\infty} (-1)^k \dim \mathcal{H}^k(A)$$

has the form

$$\chi(A) = \chi_B \left(\sum_{t=0}^{\infty} (-1)^t \dim H^t(\mathfrak{h}_x) \right),$$

where χ_B is the Euler characteristic of the manifold B.

Let us consider some simple examples.

EXAMPLE 4.1. Suppose that the Lie algebra $\mathfrak{g}_x$ is *reductive*, that is, $\mathfrak{g}_x = z\mathfrak{g}_x \oplus [\mathfrak{g}_x, \mathfrak{g}_x]$, where $\mathfrak{h}_x = [\mathfrak{g}_x, \mathfrak{g}_x]$ is a semisimple Lie algebra. By (3.6), each summand in the direct sum of vector bundles $\mathfrak{g} = z\mathfrak{g} \oplus [\mathfrak{g}, \mathfrak{g}]$ is ad-invariant. Therefore, an Abelian algebroid $(A_0 \to B, q, \{\ ,\ \}_0)$ with $A_0 = \mathcal{P} \oplus z\mathfrak{g} \simeq TB \oplus z\mathfrak{g}$ is defined. Since the Lie algebra $\mathfrak{h}_x$ is semisimple, by the Whitehead lemma [**CE**] we have $H^1(\mathfrak{h}_x) = H^2(\mathfrak{h}_x) = 0$. Hence, if we consider the spectral sequence related to the subbundle $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$, we obtain $E_2^{s,1} = E_2^{s,2} = 0$. Thus, we have proved (see formula (4.16)) that the first cohomology spaces have the form

$$\begin{aligned}
\mathcal{H}^1(A) &\simeq \mathcal{H}^1(A_0), \\
\mathcal{H}^2(A) &\simeq \mathcal{H}^2(A_0), \\
\mathcal{H}^3(A) &\simeq \mathcal{H}^3(A_0) \oplus \operatorname{Ker} d_4^{0,3},
\end{aligned}$$

where $\mathcal{H}^k(A_0)$ is the cohomology space of an Abelian algebroid. We calculate the cohomology of an Abelian algebroid in the next section.

EXAMPLE 4.2. Let the base B of a quasiparallelizable Lie algebroid have the de Rham cohomology of spherical type, i.e.,

$$\dim H^s(B) = \begin{cases} 1, & s = 0, n, \\ 0, & s \neq 0, n, \end{cases} \qquad n \geq 2.$$

Consider the spectral sequence related to the subbundle $\mathfrak{h} = \mathfrak{g}$. Formula (4.21) has the form

$$E_2^{s,t} = \begin{cases} H^t(\mathfrak{g}_x), & s = 0, n, \\ 0, & s \neq 0, n. \end{cases}$$

Hence $E_n^{s,t} = E_2^{s,t}$, and the operator $d_n^{0,k}\colon E_n^{0,k} \to E_n^{k,k-n+1}$ defines a linear mapping

$$\varpi_k\colon \mathbf{h}^k \to \mathbf{h}^{k-n+1}, \tag{4.23}$$

where for $k < 0$ the cohomology spaces $\mathbf{h}^k \overset{\text{def}}{=} H^k(\mathfrak{g}_x)$ of the Lie algebra $\mathfrak{g}_x$ are supplemented by zeros. Obviously, the spectral sequence degenerates in the $(n+1)$th term $E_{n+1}^{s,t} = \operatorname{Ker} d_n^{s,t} / \operatorname{Ran} d_n^{s+n,t-n+1}$. Therefore, we have the following formulas for the cohomology of the considered Lie algebroid:

$$\mathcal{H}^k(A) \simeq \mathbf{h}^k \qquad \forall\, k = 0, 1, \dots, n-2, \tag{4.24}$$

$$\mathcal{H}^{n-1}(A) \simeq \operatorname{Ker} \varpi_{n-1}, \tag{4.25}$$

$$\mathcal{H}^k(A) \simeq (\operatorname{Ker} \varpi_k) \oplus (\mathbf{h}^{k-n} / \operatorname{Ran} \varpi_{k-1}), \qquad k \geq n. \tag{4.26}$$

EXAMPLE 4.3. Suppose that the base of the considered transitive Lie algebroid is simply connected ($\pi_1(B) = 0$). In this case A is quasiparallelizable and $E_2^{1,t} = 0$. Therefore, we have a linear mapping

$$\mathcal{E}_k = d_2^{0,k}\colon \mathbf{h}^k \to \mathbf{h}^{k-1} \otimes H^2(B),$$

and $\mathcal{H}^1(A) \simeq \operatorname{Ker} \mathcal{E}_1$. If in addition $H^3(B) = 0$, then

$$\mathcal{H}^2(A) \simeq \operatorname{Ker} \mathcal{E}_2 \oplus \big(H^2(B) / \operatorname{Ran} \mathcal{E}_1\big).$$

EXAMPLE 4.4. Suppose that the Lie algebroid A admits a flat adjoint connection $\nabla^{\mathcal{P}}$. Also suppose that there exists an ad-invariant subbundle $\mathfrak{h}$, transversal to $z\mathfrak{g}$ ($\mathfrak{g} = z\mathfrak{g} \oplus \mathfrak{h}$). Consider operators (4.7)–(4.10) in $K^{s,t}$ (4.2), corresponding to the decomposition $A = \mathcal{P} \oplus z\mathfrak{g} \oplus \mathfrak{h}$. Then the following equalities hold:

$$\delta = 0, \qquad d_1^2 = 0.$$

In other words, the coboundary operator (4.6) is a sum of two coboundary operators d_0 and d_1; the calculation of cohomology for such an operator falls in the well-known procedure for the spectral sequence of a double complex (see, e.g., [**BT**]).

If in addition we assume that the holonomy group of this flat connection is trivial (for example, B is simply connected), then it is easy to see that the spectral sequence degenerates at $E_2^{s,t}$ (i.e. $d_r^{s,t} = 0\ \forall r \geq 2$), and

$$\mathcal{H}^k(A) \simeq \bigoplus_{t+s=k} \mathbf{h}^t \otimes \mathcal{H}^s(A_0).$$

In some applications (see, for example, [**KM**]) it is important to describe the cocycles $Z^k(A)$ of the operator D. We give some formulas for $Z^1(A)$ and $Z^2(A)$.

By $Z^{s,t}_{d_0}$ ($B^{s,t}_{d_0}$) and $Z^{s,t}_{d_1}$ ($B^{s,t}_{d_1}$) we denote the kernels (images) of the operators d_0 and d_1 respectively. Let us fix a linear mapping $d_1^{-1}\colon B^{s+1,t}_{d_1} \to \mathcal{K}^{s,t}$ such that $d_1 d_1^{-1} = id_{B^{s+1,t}_{d_1}}$. Let us consider the following linear spaces:

$$L_0^{s,t} \overset{\text{def}}{=} \{\eta \in Z^{s,t}_{d_0} \mid d_1\eta \in B^{s,t}_{d_0}\}, \tag{4.27}$$

$$L_1^{s,t} \overset{\text{def}}{=} \{\eta \in L_0^{s,t} \mid \delta\eta \in B^{s+2,t-1}_{d_1} \text{ and } d_0 d_1^{-1}\delta\eta = d_1\eta\}, \tag{4.28}$$

$$L_2^{0,2} \overset{\text{def}}{=} \{\eta \in L_1^{0,2} \mid \delta d_1^{-1}\delta\eta \in B^{3,0}_{d_1}\}. \tag{4.29}$$

PROPOSITION 4.12. *There exist isomorphisms*

$$i_1\colon L_1^{0,1} \oplus L_0^{1,0} \xrightarrow{\simeq} Z^1(A),$$
$$i_2\colon L_2^{0,2} \oplus L_1^{1,1} \oplus L_0^{2,0} \xrightarrow{\simeq} Z^2(A)$$

that are given by the formulas

$$i_1(\eta^{0,1} + \eta^{1,0}) = e^{0,1}\eta^{0,1} + e^{1,0}(\eta^{1,0} - d_1^{-1}\delta\eta^{0,1}), \tag{4.29}$$

$$\begin{aligned} i_2(\eta^{0,2} + \eta^{1,1} + \eta^{2,0}) &= e^{0,2}\eta^{0,2} + e^{1,1}(\eta^{1,1} - d_1^{-1}\delta\eta^{0,2}) \\ &\quad + e^{2,0}(\eta^{2,0} - d_1^{-1}\delta\eta^{1,1} + d_1^{-1}\delta d_1^{-1}\delta\eta^{0,2}) \end{aligned} \tag{4.30}$$

(*here* $\eta^{s,t} \in L_t^{s,t}$).

The proof is a straightforward verification.

§5. Abelian transitive algebroids

We shall say that two Lie algebroids are isomorphic: $(A_1 \to B, q_1, \{\,,\,\}_1) \overset{f}{\simeq} (A_2 \to B, q_2, \{\,,\,\}_2)$ if there exists an isomorphism of vector bundles $f\colon A_1 \to A_2$ such that $q_2 \circ f = q_1$ and

$$f\{\alpha, \beta\}_1 = \{f\alpha, f\beta\}_2 \qquad \forall\, \alpha, \beta \in \Gamma(A_1). \tag{5.0}$$

Let $(A \to B, q, \{\,,\,\})$ be an Abelian transitive algebroid. As pointed out in Corollary 3.4, the adjoint connection in the Abelian algebroid is flat and independent of the choice of the transversal subbundle $\mathcal{P}$; we denote the adjoint connection by ∇. Note that if we obtain an Abelian transitive algebroid by restriction of the algebroid for a regular Poisson manifold, then the adjoint connection is a Bott-type connection [**Bo**$_1$, **Li**$_3$] associated to the symplectic foliation.

By choosing a transversal subbundle $\mathcal{P}$ (3.8), we obtain an isomorphism of Lie algebroids $(A \to B, q, \{\,,\,\}) \overset{f_{\mathcal{P}}}{\simeq} (TB \oplus \mathfrak{g} \to B, q_1, \{\,,\,\}_{R^{\mathcal{P}}})$, where q_1 is the projection of $TB \oplus \mathfrak{g}$ on the first summand, $f_{\mathcal{P}}\alpha = q\alpha \oplus (\alpha - pq\alpha)$, and the bracket $\{\,,\,\}_{R^{\mathcal{P}}}$ is defined as follows:

$$\{u_1 \oplus \theta_1, u_2 \oplus \theta_2\}_{R^{\mathcal{P}}} = [u_1, u_2] \oplus \big(R^{\mathcal{P}}(u_1, u_2) + \nabla_{u_1}\theta_2 - \nabla_{u_2}\theta_1\big). \tag{5.1}$$

By $Z^s_\nabla(\mathfrak{g})$ and $H^s_\nabla(\mathfrak{g})$ we denote the cocycles and cohomology of the differential (A.4) corresponding to the adjoint connection in the vector bundle $\mathfrak{g}$ (see §7).

LEMMA 5.1. (a) $R^{\mathcal{P}} \in Z^2_\nabla(\mathfrak{g})$.

(b) *The cohomology class* $[R^{\mathcal{P}}] \in H^2_\nabla(\mathfrak{g})$ *is independent of the choice of a transversal subbundle* $\mathcal{P}$.

PROOF. (a). The equality $\nabla R^{\mathcal{P}} = 0$ follows from the Jacoby identity for the bracket (5.1) and coincides with the Bianchi identity for the connection in the Lie algebroid in the sense of Mackenzie.

(b). Suppose that $\mathcal{P}$ and $\mathcal{P}'$ are two transversal subbundles (3.8), and p and p' are the corresponding morphisms (3.9). We define $T \in \Gamma(T^*B \otimes \mathfrak{g})$ as $T(u) \stackrel{\text{def}}{=} pu - p'u$ $(u \in \mathcal{V}^1(B))$. Then

$$\begin{aligned} R^{\mathcal{P}}(u_1, u_2) - R^{\mathcal{P}'}(u_1, u_2) &= \{pu_1, pu_2\} - p[u_1, u_2] - \{p'u_1, p'u_2\} + p'[u_1, u_2] \\ &= \{pu_1, pu_2 - p'u_2\} + \{pu_1 - p'u_1, p'u_2\} - T([u_1, u_2]) \\ &= \nabla_{u_1} T(u_2) - \nabla_{u_2} T(u_1) - T([u_1, u_2]) = \nabla T(u_1, u_2). \end{aligned}$$

Hence, we have proved that $R^{\mathcal{P}} = R^{\mathcal{P}'} + \nabla T$. The lemma is proved. □

The above lemma states that the structure of an Abelian algebroid uniquely determines a certain cohomology class in $H^2_\nabla(\mathfrak{g})$. The converse statement also holds. Suppose that the vector bundle $\mathcal{E} \to B$ has a flat connection ∇. By $\mathrm{Iso}_\nabla(\mathcal{E})$ we denote the group of isomorphisms of the vector bundle $\mathcal{E}$, preserving the connection ∇:

$$\mathrm{Iso}_\nabla(\mathcal{E}) = \{F \in \Gamma(GL(\mathcal{E})) \mid \nabla_u F\theta = F\nabla_u \theta \ \ \forall u \in \mathcal{V}^1(B), \theta \in \Gamma(\mathcal{E})\}.$$

The group $\mathrm{Iso}_\nabla(\mathcal{E})$ acts on the space $H^s_\nabla(\mathcal{E})$ in the natural way: $F[R] = [FR]$.

THEOREM 5.2. (a) *Each section* $R \in Z^2_\nabla(\mathcal{E})$ *determines the structure of an Abelian algebroid* $(TB \oplus \mathcal{E} \to B, q, \{\,,\,\}_R)$, *where* q *is the projection on the first summand and the bracket* $\{\,,\,\}_R$ *is defined by* (5.1).

(b) *The space of nonisomorphic structures of Abelian algebroids on* $A = TB \oplus \mathcal{E}$ *with adjoint connection* ∇ *coincides with the space of orbits for the action of the group* $\mathrm{Iso}_\nabla(\mathcal{E})$ *in the space* $H^2_\nabla(\mathcal{E})$.

PROOF. Statement (a) is verified by direct calculations; it is an Abelian version of a more general statement that can be found in [**Mz**].

(b) Let us consider two Abelian algebroids $(TB \oplus \mathcal{E} \to B, q, \{\,,\,\}_1)$ and $(TB \oplus \mathcal{E} \to B, q, \{\,,\,\}_2)$, where q is the projection on the first summand and the brackets $\{\,,\,\}_1$, $\{\,,\,\}_2$ are defined by the formula

$$(5.1i) \qquad \{u_1 \oplus \theta_1, u_2 \oplus \theta_2\}_i = [u_1, u_2] \oplus \big(R_i(u_1, u_2) + \nabla_{u_1}\theta_2 - \nabla_{u_2}\theta_1\big),$$

where $R_i \in Z^2_\nabla(\mathcal{E})$, $i = 1, 2$. Let f be an isomorphism of the vector bundle $TB \oplus \mathcal{E}$ such that $qf = q$. We write

$$(5.2) \qquad f(u \oplus \theta) = u \oplus \big(T(u) + F\theta\big), \qquad T \in C^1(\mathcal{E}), \quad F \in \Gamma\big(GL(\mathcal{E})\big).$$

It is easy to see that, by (5.0), f in (5.2) is an isomorphism of the algebroids (5.1i) if and only if the following two conditions are satisfied:

$$(5.3) \qquad F\nabla_u \theta = \nabla_u F\theta \qquad \forall u \in \mathcal{V}^1(B), \quad \theta \in \Gamma(\mathcal{E});$$

$$(5.4) \qquad R_2(u_2, u_1) = FR_1(u_2, u_1) + \nabla T(u_1, u_2).$$

Formula (5.3) means that $F \in \mathrm{Iso}_\nabla(\mathcal{E})$. Formula (5.4) means that the cohomology classes $[R_1], [R_2] \in H^2_\nabla(\mathcal{E})$ lie on the same orbit of the action of the group $\mathrm{Iso}_\nabla(\mathcal{E})$. The theorem is proved. □

COROLLARY 5.3. *Let the base B be simply connected. Then the space of non-isomorphic structures of an Abelian algebroid on the bundle $TB \oplus \mathbb{R}^n$ is isomorphic to the space*

$$(H^2(B) \otimes \mathbb{R}^n)/GL(n) \simeq G_0(H^2(B)) \sqcup G_1(H^2(B)) \sqcup \cdots \sqcup G_n(H^2(B)), \tag{5.5}$$

where $G_k(V)$ is the space of k-dimensional planes in the linear space V and $\sqcup$ is the disjoint union.

REMARK 5.4. In fact, Corollary 5.3 says that up to an isomorphism the structure of an Abelian algebroid that has a trivial holonomy group is uniquely determined by a linear subspace in $H^2(B)$.

Now we calculate the cohomology of the Abelian algebroid. Consider the spectral sequence $(E^{s,t}_r, d^{s,t}_r)$ corresponding to the subbundle $\mathfrak{g}$ (see Section 4). Then the algebroid $(A_0 \to B, q, \{\ ,\ \}_0)$ is isomorphic to the trivial algebroid $(TB \to B, \mathrm{id}_{TB}, [\,])$, and $K^{s,t} = \Gamma(\Lambda^t\mathfrak{g}^* \otimes \Lambda^s T^*B)$.

By $Z^{s,t}_\nabla$, $B^{s,t}_\nabla$, and $H^{s,t}_\nabla$, we denote the cocycles, coboundaries, and cohomology of the operator $\nabla^{s,t}$ (A.4), (4.9), corresponding to the flat connection in the bundle $\bigwedge^t \mathfrak{g}^*$, induced by the coadjoint connection. By Proposition 4.4, we have

$$(E^{s,t}_2, d^{s,t}_2) \simeq (H^{s,t}_\nabla, \widetilde{\delta}),$$

where the operator $\widetilde{\delta}\colon H^{s,t}_\nabla \to H^{s+2,t-1}_\nabla$ generated by the operator δ in (4.10) is the pairing of the cohomology class from $H^{s,t}_\nabla = H^s_\nabla(\bigwedge^t \mathfrak{g}^*)$ with the cohomology class $[R^{\mathcal{P}}] \in H^2_\nabla(\mathfrak{g})$ (cf. Theorem 8 in [**HS**]). As a corollary, we obtain the following proposition.

PROPOSITION 5.5. *If $[R^{\mathcal{P}}] = 0$, then $\mathcal{H}^k(A) = \bigoplus_{s+t=k} H^{s,t}_\nabla$.*

For each linear subspace $M \subset Z^{s,t}_\nabla$ we define

$$(\nabla^{-1}\delta)M \overset{\text{def}}{=} \{\zeta \in \mathcal{K}^{s+1,t-1} \mid \nabla^{s+1,t-1}\zeta \in \delta M\}.$$

Let us define the sequence of embedded linear spaces

$$Z^{s,t}_\nabla = L^{s,t}_0 \supset L^{s,t}_1 \supset \cdots \supset L^{s,t}_t = L^{s,t}_\infty,$$
$$L^{s,t}_r \overset{\text{def}}{=} \{\eta \in L^{s,t}_{r-1} \mid (\delta(\nabla^{-1}\delta)^{r-1}\{\eta\}) \cap B^{s+r+1,t-r}_\nabla \neq \varnothing\}, \qquad r \geq 1.$$

Since $\delta\nabla^{s,t} = \nabla^{s+2,t-1}\delta$ (see (4.14)), we obviously have $B^{s,t}_\nabla \subset L^{s,t}_\infty$ and $\delta Z^{s-2,t+1}_\nabla \subset L^{s,t}_\infty$.

THEOREM 5.6. *There exists an isomorphism of the following spaces:*

$$Z^k(A) \simeq \bigoplus_{s=0}^{k} L^{s,k-s}_{k-s}, \tag{5.6}$$

$$\mathcal{H}^k(A) \simeq L^{0,k}_k \oplus (L^{1,k-1}_{k-1}/B^{1,k-1}_\nabla) \oplus \bigoplus_{s=2}^{k} L^{s,k-s}_{k-s}/(B^{s,k-s}_\nabla + \delta Z^{s-2,k-s+1}_\nabla). \tag{5.7}$$

The proof of this theorem follows from the standard consideration (see, e.g., [**BT**]) of the spectral sequence for the double complex $(K^{s,t}, d = (-1)^t \nabla^{s,t} + \delta)$; in particular, $d_r^{s,t} = \delta(\nabla^{-1}\delta)^{r-1}$ and the isomorphism $i_k\colon \bigoplus_{s=0}^{k} L_{k-s}^{s,k-s} \xrightarrow{\simeq} Z^k(A)$ is given by the formula

$$i_k\left(\sum_{s=0}^{k} \eta^{s,k-s}\right) = \sum_{s=0}^{k} e^{s,k-s} \sum_{l=0}^{s} (-d_1^{-1}\delta)^{s-l} \eta^{l,k-l},$$

where $\eta^{s,k-s} \in L_{k-s}^{s,k-s}$ and to each $\eta^{s,k-s}$ the operator $(-d_1^{-1}\delta)^r$ assigns the last element from the chain of equalities:

$$\begin{aligned} \delta\eta^{s,k-s} &= -d_1\big((-d_1^{-1}\delta)\eta^{s,k-s}\big), \\ \delta\big((-d_1^{-1}\delta)\eta^{s,k-s}\big) &= -d_1\big((-d_1^{-1}\delta)^2\eta^{s,k-s}\big), \\ &\cdots\cdots \\ \delta\big((-d_1^{-1}\delta)^{r-1}\eta^{s,k-s}\big) &= -d_1\big((-d_1^{-1}\delta)^r\eta^{s,k-s}\big) \end{aligned}$$

REMARK. Note that as a corollary of Theorem 5.2 we obtain that the cohomology space of an Abelian transitive algebroid can be calculated using only the cohomology class $[R^{\mathcal{P}}] \in H^2_\nabla(\mathcal{E})$. In the case of a trivial holonomy group this means that the cohomology of an algebroid is calculated in terms of multiplicative properties of some subspace in $H^2(B)$. This approach has been used in a number of papers. Formulas similar to (5.6) and (5.7) for the dimensions $k = 1, 2, 3$ were obtained in [$\mathbf{VK}_{1,2}$] for some kinds of regular Poisson manifolds (see the algebroid version of these formulas in Theorem 5.7). Formulas similar to (5.7) were obtained in [**Xu**] for any k. For the spectral sequence of regular Poisson manifold see also [**Va**].

The case of a trivial holonomy group. Let us consider the case in which the holonomy group of the adjoint connection is trivial. This happens if the base of an Abelian algebroid is simply connected. In the case of an Abelian algebroid over a nondegenerate symplectic leaf $\mathcal{O}$ of a Poisson manifold, for a holonomy group to be trivial it also suffices that there exists a set of Casimir functions $k_1, k_2, \ldots, k_n$ ($n = \operatorname{codim} \mathcal{O} = \dim \mathfrak{g}_x$) in a tubular neighborhood of this leaf such that their differentials are linearly independent at each point of this leaf.

Let us choose a basis of parallel sections $X_1, X_2, \ldots, X_n \in Z^{0,1}_\nabla$, $n = \dim \mathfrak{g}_x$, such that the closed differential forms

$$\omega_j \stackrel{\text{def}}{=} \delta X_j, \qquad j = 1, 2, \ldots, n, \tag{5.8}$$

satisfy the conditions

$$\begin{gathered} [\omega_1], \ldots, [\omega_q] \quad \text{are linearly independent in } H^2(B), \\ \omega_{q+1} = d\beta_{q+1}, \omega_{q+2} = d\beta_{q+2}, \ldots, \omega_n = d\beta_n, \qquad \beta_j \in \Gamma(T^*B). \end{gathered} \tag{5.9}$$

Obviously, $\operatorname{Span}\{[\omega_j]\} \subset H^2(B)$ is a subspace that determines the structure of the considered Abelian algebroid (see Remark 5.4). In particular, if all the forms ω_j are exact ($q = 0$), i.e., if $[R^{\mathcal{P}}] = 0$, then we have $d_2^{s,t} = \delta = 0$ and $\mathcal{H}^k(A) \simeq \bigoplus_{s+t=k} (\bigwedge^t \mathfrak{g}_z) \otimes H^s(B)$.

Condition (5.9) means that $L_1^{0,1} = \operatorname{Span}\{X_j\}_{j=q+1,\ldots,n}$.

THEOREM 5.7. *Let the holonomy group of the adjoint connection in an Abelian transitive algebroid be trivial. Then*

$$\mathcal{H}^1(A) \simeq L_1^{0,1} \oplus H^1(B),$$
$$\mathcal{H}^2(A) \simeq \bigwedge^2 L_1^{0,1} \oplus W^1 \oplus \big(L_1^{0,1} \otimes H^1(B)\big) \oplus \widetilde{H}^2(B),$$

and

$$H^1(B) = 0 \quad \Longrightarrow \quad \mathcal{H}^3(A) \simeq \bigwedge^3 L_1^{0,1} \oplus (\widetilde{H}^2 \otimes L_1^{0,1}) \oplus \widetilde{W}^2 \oplus H^3(B),$$

where

$$\widetilde{H}^s(B) \stackrel{\text{def}}{=} H^s(B)/\big(\operatorname{Span}\{[\omega_j]\}_{j=1,\dots,q} \wedge H^{s-2}(B)\big),$$
$$W^s \subset (Z_\nabla^{0,1}/L_1^{0,1}) \otimes H^s(B),$$
$$W^s \stackrel{\text{def}}{=} \Big\{ \sum_{j=1}^{q} X_j \otimes [\varkappa_j] \mid [\varkappa_j] \in H^s(B), \sum_{j=1}^{q} \omega_j \wedge \varkappa_j = d\beta,\ \beta \in \Gamma(\wedge^{s+1}T^*B) \Big\},$$
$$\widetilde{W}^2 \stackrel{\text{def}}{=} W^2 / \operatorname{Span}\big\{(X_j \otimes [\omega_i] - X_i \otimes [\omega_j]) \mid i,j \le q\big\}.$$

The proof of this theorem follows from (5.7) and, in fact, is presented in [$\mathbf{VK}_1$] for the case of regular Poisson manifolds.

§6. Cohomology of homogeneous algebroids, examples

Let $\mathbb{G}$ a Lie group, $\mathbb{H}$ a closed connected subgroup, and $B \stackrel{\text{def}}{=} \mathbb{H} \setminus \mathbb{G}$ a *right homogeneous manifold.*

Let us consider a mapping $\tau^*\colon C^\infty(B) \to C^\infty(\mathbb{G})$, $\tau^*\varphi \stackrel{\text{def}}{=} \varphi \circ \tau$, where $\tau\colon \mathbb{G} \to B$ is the natural projection. Obviously, the image of τ^* is the space of functions invariant under the left action $L_h g = hg$ of $\mathbb{H}$ on the Lie group $\mathbb{G}$.

By G we denote a Lie algebra of left-invariant vector fields on the Lie group $\mathbb{G}$. We consider a homomorphism $q'\colon G \to \mathcal{V}^1(B)$,

$$\mathcal{L}_{q'u}\varphi = (\tau^*)^{-1}\mathcal{L}_u \tau^*\varphi, \qquad u \in G, \quad \varphi \in C^\infty(B). \tag{6.0}$$

This homomorphism can be extended in a natural way to the morphism of vector bundles $q\colon B \times G \to TB$, where $q\imath u = q'u$, and $\imath\colon G \to \Gamma(B \times G)$ is an isomorphism between the space of constant sections of a trivial vector bundle $A \stackrel{\text{def}}{=} B \times G \to B$ and G.

A transitive Lie algebroid $(A \to B, q, \{\,,\,\})$, where $A = B \times G$, q is defined as above, and $\{\,,\,\}$ is defined by (2.3) (see Example 2 in section 2), will be called a *homogeneous algebroid.*

In fact, the cohomology of a homogeneous algebroid is exactly the cohomology of the Lie algebra G with coefficients in the G-module $C^\infty(\mathbb{H} \setminus \mathbb{G})$. Nevertheless, we prefer to use the algebroid approach.

The following theorem gives a "lower bound" for the cohomology of a homogeneous algebroid in the case when $\mathbb{H} \setminus \mathbb{G}$ is compact.

PROPOSITION 6.1. *Suppose that $B = \mathbb{H} \setminus \mathbb{G}$ is compact and possesses a $\mathbb{G}$-invariant measure μ. Then the homomorphism $\imath\colon G \to \Gamma(A)$ induces the inclusion of the cohomology of a Lie algebra into the cohomology of a homogeneous algebroid*

$$H^\cdot(G) \hookrightarrow \mathcal{H}^\cdot(A). \tag{6.1}$$

To prove this proposition we need the following simple algebraic lemma.

LEMMA 6.2. *Let $\mathcal{K}$ be a linear space endowed with the differential d, $d^2 = 0$, and with a chain projection $I\colon \mathcal{K} \to \mathcal{K}$,*

$$I^2 = I, \qquad dI - Id = 0. \tag{6.2}$$

Then there is an exact sequence

$$0 \to H_d \operatorname{Ran} I \to H_d \mathcal{K} \to H_d \operatorname{Ker} I \to 0,$$

where H_d denotes the cohomology of d.

PROOF OF PROPOSITION 6.1. Let us consider an operator $\imath_*\colon C^k(G) \to C^k(A)$ such that $(\imath_* c)(\imath u_1, \dots, \imath u_k) = c(u_1, u_2, \dots, u_k)$. Obviously, $\imath_*$ is a chain mapping: $D\imath_* = \imath_* d_G$, where d_G is the standard differential in $C^\cdot(G) \oplus_k \bigwedge^k G^*$.

Let us also introduce an averaging operator $I_\mu\colon C^k(A) \to C^k(A)$ by the formula

$$(I_\mu f)(\imath u_1, \dots, \imath u_k) = \int_B f(\imath u_1, \dots, \imath u_k)\mu, \qquad f \in C^k(A), \quad u_j \in G.$$

The image of I_μ is exactly the image of the inclusion $\imath_*$, so that the operator $\imath_*$ induces an isomorphism of $H^k(G)$ and the cohomology of $\operatorname{Ran} I_\mu$ with the differential D (2.4).

Since B is compact, we may suppose that $\int_B \mu = 1$, and therefore $I_\mu^2 = I_\mu$. Since μ is invariant under the action of $\mathbb{G}$, we have

$$\int_B \mathcal{L}_{q'u}(\varphi)\mu = 0 \qquad \forall u \in G, \quad \forall \varphi \in C^\infty(B),$$

and therefore

$$\begin{aligned}(DI_\mu f - I_\mu D f)&(\imath u_1, \dots, \imath u_k) \\ &= -\sum_{j=0}^{k} (-1)^j \int_B \mathcal{L}_{q'u_j}\big(f(\imath u_o, \imath u_1, \dots, \widehat{\imath u_j}, \dots \imath u_k)\big)\mu = 0.\end{aligned}$$

By applying Lemma 6.2 to the chain projection I_μ, we complete the proof. □

The following proposition is somewhat dual to Proposition 6.2.

PROPOSITION 6.3. *Suppose that $\mathbb{H}$ is compact. Then there is an isomorphism of Grassmann algebras*

$$\mathcal{H}^\cdot(A) \simeq H^\cdot(\mathbb{G}), \tag{6.3}$$

where $H^\cdot(\mathbb{G})$ is the de Rham cohomology of $\mathbb{G}$.

Although the proof of this proposition can be derived from [**Mz**] or, probably, from another source unknown to the authors, we give a brief sketch of the proof.

PROOF. Let us define the inclusion mapping $\tau_k^*\colon C^k(A) \to \Omega^k(\mathbb{G})$ of $C^k(A)$ into the space of differential forms on the group $\mathbb{G}$ by the formula

$$(\tau_k^* f)(u_1, \dots, u_k) = \tau^*\big(f(\imath u_1, \dots, \imath u_k)\big), \qquad u_j \in G, \quad f \in C^k(A).$$

The image of τ_k^* is the space of differential forms $\Omega_L^k(\mathbb{G}, \mathbb{H})$ that are invariant under the left action of the subgroup $\mathbb{H}$

$$\operatorname{Ran} \tau_k^* = \Omega_L^k(\mathbb{G}, \mathbb{H}) \stackrel{\text{def}}{=} \{\omega \in \Omega^k(\mathbb{G}) \mid L_h^* \omega = \omega \; \forall \, h \in \mathbb{H}\}. \tag{6.4}$$

The inclusion mapping τ_k^* satisfies the conditions

$$\tau_{k+1}^* D = d\tau_k^*,$$
$$\tau_{k_1+k_2}^* f_1 \wedge f_2 = \tau_{k_1}^* f_1 \wedge \tau_{k_2}^* f_2, \qquad f_i \in C^{k_i}(A), \quad i = 1, 2,$$

and thus, defines an isomorphism between Grassmann algebras $\mathcal{H}^k(A)$ and the de Rham cohomology $H_L^k(\mathbb{G}, \mathbb{H})$ of $\mathbb{H}$-left-invariant differential forms.

By μ_h we denote the Haar measure on a compact group $\mathbb{H}$ such that $\int \mu_h = 1$. Introduce an operator $I_{\mathbb{H}} \colon \Omega^k(\mathbb{G}) \to \Omega^k(\mathbb{G})$:

$$I_{\mathbb{H}} \omega \stackrel{\text{def}}{=} \int L_h^* \omega \mu_h.$$

The image of $I_{\mathbb{H}}$ is exactly the space of $\mathbb{H}$-left-invariant differential forms (6.4). It can be easily verified that $I_{\mathbb{H}}$ satisfies conditions (6.2). Applying Lemma 6.2, it remains to prove that each closed differential form that lies in $\operatorname{Ker} I_{\mathbb{H}}$ is exact.

Let $[\omega] \in H^k(\mathbb{G})$, $I_{\mathbb{H}}\omega = 0$. Since the subgroup $\mathbb{H}$ is connected, the diffeomorphism $L_h \colon \mathbb{G} \to \mathbb{G}$ is homotopic to the identity, and thus, generates an identical mapping in $H^k(\mathbb{G})$:

$$L_h^* \omega = \omega + d\widetilde{\omega}_h.$$

Since $\omega = \operatorname{Ker} I_{\mathbb{H}}$, we have

$$0 = \int L_h^* \omega \mu_h = \int (\omega + d\widetilde{\omega}_h) \mu_h = \omega + d \int \widetilde{\omega}_h \mu_h;$$

hence, $\omega = -d \int \widetilde{\omega}_h \mu_h$. This completes the proof. □

REMARK 6.4. Suppose that the group $\mathbb{G}$ is compact. If we consider the subgroup $\mathbb{H} = \mathbb{G}$ (i.e., the case in which B is a point), we obtain the classical result $H^{\cdot}(\mathbb{G}) = H^{\cdot}(G)$. For another closed subgroup $\mathbb{H} \subset \mathbb{G}$ we obtain the transitive case of the result proved in [**GW**]: $\mathcal{H}^{\cdot}(A) = H^{\cdot}(G)$.

Let us consider the coadjoint action of a Lie group $\mathbb{G}$ on its coalgebra G^*. On the Euclidean space G^* a *linear Poisson bracket* is defined (see, e.g., [**Ki**$_1$]) and the Lie algebroid $(G \times G^* \to G^*, q, \{\,,\,\})$, which can be constructed by this Poisson bracket, falls into both examples of Section 2. For a tame orbit $\mathcal{O}$ let us consider a reduced homogeneous algebroid $(G \times \mathcal{O} \to \mathcal{O}, q, \{\,,\,\})$. Each orbit $\mathcal{O}$ of a coadjoint action possess the invariant measure $\mu = \underbrace{\omega_{\mathcal{O}} \wedge \omega_{\mathcal{O}} \wedge \cdots \wedge \omega_{\mathcal{O}}}_{\frac{1}{2}(\dim \mathcal{O}) \text{ times}}$ which is the symplectic volume corresponding to the Kirillov symplectic form $\omega_{\mathcal{O}}$ (see [**Ki**$_1$]). Therefore, for each homogeneous algebroid over the compact orbit of coadjoint action the assumptions of Proposition 6.1 are satisfied.

Poisson cohomology of transitive Lie algebroids over $e(3)^*$. Below we calculate the cohomology for homogeneous algebroids over the orbits of the coadjoint action of the group $E(3)$. As will be shown, without the assumption that the group $\mathbb{G}$ is compact, homogeneous algebroids over orbits of different types may have different cohomology. This example also shows that without any additional assumptions the inclusion (6.1) need not be an isomorphism.

Let us consider a Lie group $\mathbb{G} = E(3)$ of motions in the Euclidean space $\mathbb{R}^3$. In $\mathcal{N} = e(3)^*$ we introduce coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ so that they satisfy the following commutation relations under the linear Poisson bracket:

$$\begin{aligned}
&\{y_i, y_j\} = 0, && i, j = 1, 2, 3; && \\
&\{x_1, x_2\} = x_3, && \{x_2, x_3\} = x_1, && \{x_3, x_1\} = x_2, \\
&\{x_1, y_2\} = y_3, && \{x_2, y_3\} = y_1, && \{x_3, y_1\} = y_2,
\end{aligned}$$

The linear Poisson bracket in $\mathcal{N} = e(3)^*$ has two Casimir functions:

$$k_1 = y_1^2 + y_2^2 + y_3^2, \qquad k_2 = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

There are two kinds of orbits of coadjoint action in $e(3)^*$ (see, e.g., [**MR**]):
(i) *nondegenerate orbits* $\mathcal{O}_{k_1,k_2} = \{k_1 = \text{const} \neq 0, k_2 = \text{const}\} \simeq TS^2$;
(ii) *degenerate orbits* $\mathcal{O}_\rho = \{x_1^2 + x_2^2 + x_3^2 = \rho^2 \neq 0, y_1 = y_2 = y_3 = 0\} \simeq S^2$.

Nondegenerate case. Let us calculate the cohomology of the homogeneous algebroid over a nondegenerate orbit $\mathcal{O}_{k_1,k_2}$. Following [**VK**$_2$] we introduce vector fields $u_1, u_2 \in \mathcal{V}^1(e(3)^*)$ such that

$$\mathcal{L}_{u_j}(k_i) = \delta_{ji} \tag{6.5}$$

(here δ_{ij} is the Kronecker symbol). By (6.5) we have $X_j \overset{\text{def}}{=} r^*_{\mathcal{O}_{k_1,k_2}} u_j \in \Gamma(\mathfrak{g}^*)$ of vector fields u_j to the base $\mathcal{O}_{k_1,k_2}$ gives the basis of parallel sections in the bundle $\mathfrak{g}^*$. The differential forms ω_j (5.8) can be determined from the relation $Du_j = q^*\widetilde{\omega}_j$, where D is the Lichnerowicz differential (1.6) on the Poisson manifold $\mathcal{N} = e(3)^*$, $q^* \colon \bigwedge^2 T^*\mathcal{N} \to \bigwedge^2 T\mathcal{N}$ is defined by the morphism q (1.1), and the natural restriction $r \colon \Gamma(\bigwedge^k T^*\mathcal{N}) \to \Gamma(\bigwedge^k T\mathcal{O}_{k_1,k_2})$ gives the closed differential forms $\omega_j = r\widetilde{\omega}_j$.

Since the homogeneous algebroid over a nondegenerate orbit is Abelian, to calculate its cohomology we only need to determine the subspace $\operatorname{Span}\{[\omega_j]\}$ in the one-dimensional space $H^2(\mathcal{O}_{k_1,k_2})$. The vector fields u_1 and u_2 (6.5) are given by the formulas

$$\begin{aligned}
u_1 &= \sum_{j=1}^{3} \left(\frac{1}{2k_1} x_j - \frac{k_2}{(k_1)^2} y_j \right) \frac{\partial}{\partial x_j} + \frac{1}{2k_1} \sum_{i=1}^{3} y_i \frac{\partial}{\partial y_i}, \\
u_2 &= \frac{1}{k_1} \sum_{j=1}^{3} y_j \frac{\partial}{\partial x_j}.
\end{aligned}$$

It is easy to calculate that

$$Du_2 = \frac{2}{k_1} \left(y_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + y_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + y_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right) = q^*\widetilde{\omega}_2,$$

where $\widetilde{\omega}_2 = \frac{2}{k_1^2}(y_1\,dy_2 \wedge dy_3 + y_2\,dy_3 \wedge dy_1 + y_3\,dy_1 \wedge dy_2)$ and the restriction $\omega_2 = r\widetilde{\omega}_2 \in \Gamma(\wedge^2 \mathcal{O}_{k_1,k_2})$ gives the basic form in $H^2(\mathcal{O}_{k_1,k_2})$. Therefore, the considered Lie algebroid falls in the nontrivial type of the only two possible types of Abelian algebroids over TS^2. The cohomology of this Lie algebroid is easily calculated by using either Theorems 5.7 and 5.6 or Example 4.2. The *Betti numbers*

$$b_k \stackrel{\text{def}}{=} \dim \mathcal{H}^k(A)$$

of this Lie algebroid are

$$b_1 = b_3 = b_4 = 1, \qquad b_2 = b_5 = b_6 = 0. \tag{6.6}$$

Degenerate case. Let us consider a homogeneous algebroid over the degenerate orbit $\mathcal{O}_\rho = \{x_1^2 + x_2^2 + x_3^2 = \rho^2 \neq 0,\ y_1 = y_2 = y_3 = 0\}$. It should be noted that despite the fact that the vector bundle $\mathfrak{g}$ is trivial ($\mathfrak{g}$ has a basis of parallel sections $\{dy_j,\ j = 1,2,3,\ \sum_{j=1}^3 x_j\,dx_j\}$), $\mathfrak{g}$ does not admit a flat adjoint connection. This follows from a fact that an invariant subbundle $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$ is isomorphic (as a vector bundle) to $TS^2 \to S^2$, and thus, cannot have a flat linear connection.

We introduce a transversal subbundle $\mathcal{P}$ (3.8),

$$\mathcal{P}_x \stackrel{\text{def}}{=} \left\{\beta_x \in T_x^*\mathcal{N} \mid \left\langle \beta_x, \frac{\partial}{\partial y_j}\right\rangle = 0,\ j = 1,2,3; \quad \left\langle \beta_x, \sum_{j=1}^3 x_j \frac{\partial}{\partial x_j}\right\rangle = 0\right\} \tag{6.7}$$

for $x \in \mathcal{O}_\rho$. The curvature of this transversal subbundle has the form

$$R^{\mathcal{P}}(u_1, u_2) = -\frac{1}{\rho^2}\omega_\rho(u_1,u_2)\, r_{\mathcal{O}_\rho}\Big(\sum_{j=1}^3 x_i\,dx_j\Big),$$

where $\omega_\rho = \frac{1}{\rho^2}(x_1\,dx_2 \wedge dx_3 + x_2\,dx_3 \wedge dx_1 + x_3\,dx_1 \wedge dx_2)$ is the Kirillov symplectic form of the orbit $\mathcal{O}_\rho$; it has a nonzero cohomology class in $H^2(\mathcal{O}_\rho)$. We also introduce sections $Y_0, Y_1 \in \Gamma(\mathfrak{g}^*)$, $\eta \in \Gamma(\wedge^2 \mathfrak{g}^*)$,

$$Y_0 \stackrel{\text{def}}{=} r^*_{\mathcal{O}_\rho} \sum_{j=1}^3 x_j \frac{\partial}{\partial x_j}, \qquad Y_1 \stackrel{\text{def}}{=} r^*_{\mathcal{O}_\rho} \sum_{j=1}^3 x_j \frac{\partial}{\partial y_j}, \tag{6.8}$$

$$\eta \stackrel{\text{def}}{=} r^*_{\mathcal{O}_\rho}\left(x_1 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3} + x_2 \frac{\partial}{\partial y_3} \wedge \frac{\partial}{\partial y_1} + x_3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}\right). \tag{6.9}$$

It is easy to calculate that these sections are parallel with respect to the connection induced by the adjoint connection $\nabla^{\mathcal{P}}$ (6.7).

We consider the spectral sequence that corresponds to the subbundle $\mathfrak{g}$ ($K^{s,t} = \Gamma(\bigwedge^t \mathfrak{g}^* \otimes \bigwedge^s T^*\mathcal{O}_\rho)$). The Lie algebra $\mathfrak{g}_x$ is isomorphic to $e(2) \oplus \mathbb{R}^1$ [**MR**], and it turns out that the cohomology space $E_1^{0,t} = H_{d_0}\Gamma(\wedge^t \mathfrak{g})$ can be written via parallel sections (6.8), (6.9):

$$\begin{aligned}
E_1^{0,1} &= \mathrm{Span}(Y_0, Y_1),\\
E_1^{0,2} &= \mathrm{Span}([\eta]_{d_0}, [Y_0 \wedge Y_1]_{d_0}),\\
E_1^{0,3} &= \mathrm{Span}([\eta \wedge Y_0]_{d_0}, [\eta \wedge Y_1]_{d_0}),\\
E_1^{0,4} &= \mathrm{Span}([\eta \wedge Y_1 \wedge Y_0]_{d_0}).
\end{aligned}$$

This Lie algebroid falls in the case of Example 4.2. Therefore, $E_2^{s,t} = E_1^{s,t}$, and to calculate the cohomology space $\mathcal{H}^k(A)$ it remains to obtain the linear mapping ϖ (4.23) that is defined by $d_2^{0,t}$. Since all cohomology classes of $E_2^{s,t}$ can be represented via parallel sections, the mapping $d_2^{s,t}$ (4.18) coincides with δ (4.10) on such sections. It is easy to calculate that the following equalities hold:

$$d_2^{0,1}[Y_0] = [\omega_\rho], \qquad d_2^{0,1}[Y_1] = 0,$$
$$d_2^{0,2}[\eta] = 0, \qquad d_2^{0,2}[Y_0 \wedge Y_1] = -Y_1 \boxtimes [\omega_\rho],$$
$$d_2^{0,3}[Y_0 \wedge \eta] = [\eta] \boxtimes [\omega_\rho], \qquad d_2^{0,3}[Y_1 \wedge \eta] = 0,$$
$$d_2^{0,4}[\eta \wedge Y_1 \wedge Y_0] = -[\eta \wedge Y_1] \boxtimes [\omega_\rho].$$

These equalities give us the mapping (4.23), and thus, the cohomology spaces (4.24)–(4.26). We have the following Betti numbers of the Lie algebroid over $\mathcal{O}_\rho$:

$$b_1 = b_2 = b_4 = b_5 = b_6 = 1, \qquad b_3 = 2.$$

§7. Formal equivalence

Let $\mathcal{O}$ be a symplectic manifold, $\mathcal{E} \to \mathcal{O}$ a vector bundle, $p\colon \mathcal{E}^* \to \mathcal{O}$ its dual, and $o\colon \mathcal{O} \to \mathcal{E}^*$ the zero section.

We say that a Poisson bracket on the total manifold $\mathcal{E}^*$ is *proper* if the zero section $o\colon \mathcal{O} \to \mathcal{E}^*$ is a Poisson mapping. In other words, a Poisson bracket on $\mathcal{E}^*$ is proper if the image of the zero section $\mathcal{O} \hookrightarrow \mathcal{E}^*$ is a symplectic leaf, and the induced Poisson bracket on $\mathcal{O}$ coincides with the Poisson bracket of the symplectic structure on $\mathcal{O}$.

Proper Poisson brackets arise when we consider the geometry in the vicinity of a (possibly degenerate) symplectic leaf $\mathcal{O}$ of a Poisson manifold $\mathcal{N}$. We can consider a diffeomorphism between a tubular neighborhood of $\mathcal{O}$ in $\mathcal{N}$ and the normal vector bundle $\mathcal{E}^* = T_{\mathcal{O}}\mathcal{N}/T\mathcal{O}$ (note that this diffeomorphism is not unique). The pullback of the initial Poisson bracket to $\mathcal{E}^*$ is a proper Poisson bracket.

By μ we denote an ideal of the commutative ring $C^\infty(\mathcal{E}^*)$ of functions that vanish on the submanifold $\mathcal{O} \hookrightarrow \mathcal{E}^*$. Consider a filtration of ideals

$$C^\infty(\mathcal{E}^*) = \mu^0 \supset \mu^1 \supset \mu^2 \supset \mu^3 \supset \cdots \supset \mu^\infty = \bigcap_{k=0}^{\infty} \mu^k, \tag{7.1}$$

where $\mu^k = \mu\mu^{k-1}$ is an ideal of functions that have k zero terms in the Taylor expansion along the fiber coordinates on $\mathcal{E}^*$. Considering $\mathcal{V}^\cdot(\mathcal{E}^*) = \bigoplus_k \mathcal{V}^k(\mathcal{E}^*)$ as a module over the ring $C^\infty(\mathcal{E}^*)$, we also can establish a filtration

$$\mathcal{V}^k(\mathcal{E}^*) \supset \mu^1\mathcal{V}^k(\mathcal{E}^*) \supset \mu^2\mathcal{V}^k(\mathcal{E}^*) \supset \cdots \supset \mu^\infty\mathcal{V}^k(\mathcal{E}^*).$$

In what follows we are interested only in the behavior of all considered objects in an open small neighborhood of the symplectic leaf $\mathcal{O}$. We shall call $g\colon U \to g(U)$ a *local diffeomorphism* if U is an open neighborhood in $\mathcal{E}^*$ that contains $\mathcal{O}$ and the diffeomorphism g satisfies the condition $g(\mathcal{O}) = \mathcal{O}$. Local diffeomorphisms form a group.

We shall say that two proper Poisson brackets given by Poisson tensors $\Psi, \Psi_1 \in \mathcal{V}^2(\mathcal{E}^*)$ are *formally equivalent* if for any $p \geq 2$, there exists a local diffeomorphism

g such that $g|_{\mathcal{O}} = \mathrm{id}_{\mathcal{O}}$ and

$$g_*\Psi - \Psi_1 \in \mu^p \mathcal{V}^2(\mathcal{E}^*). \tag{7.2}$$

Let $\Psi \in \mathcal{V}^2(\mathcal{E}^*)$ be a Poisson tensor of a proper Poisson bracket on $\mathcal{E}^*$. Let $(T^*_{\mathcal{O}}\mathcal{E}^* \to \mathcal{O}, \{\,,\,\}, q)$ be the corresponding transitive Lie algebroid. Note that the zero section of $\mathcal{E}^*$ induces a natural isomorphism $T_{\mathcal{O}}\mathcal{E}^* = T\mathcal{O} \oplus \mathcal{E}^*$, where $\mathcal{E}^*$ is identified with its vertical subbundle in $T\mathcal{E}^*$ restricted on $\mathcal{O}$. Therefore, we also have a natural isomorphism $T^*_{\mathcal{O}}\mathcal{E}^* = T^*\mathcal{O} \oplus \mathcal{E}$. It is easy to see that the properness of the considered Poisson bracket implies that $\mathfrak{g}_x = \operatorname{Ker} q|_x = \mathcal{E}_x$ $(x \in \mathcal{O})$, and in what follows we shall always identity $\mathcal{E}$ and $\mathfrak{g}$.

Let $\rho^p_\alpha \colon \Gamma(S^p\mathcal{E}) \to \Gamma(S^p\mathcal{E})$ be a representation of the algebroid $(T^*_{\mathcal{O}}\mathcal{E}^*, \{\,,\,\}, q)$ in the symmetric product of the vector bundle $\mathcal{E}$. This representation can be given by the following formulas (here we consider sections of $\mathcal{E}$ as sections of $\mathfrak{g} \subset T^*_{\mathcal{O}}\mathcal{E}^*$):

$$\begin{aligned} \rho^0_\alpha f &= \mathcal{L}_{q\alpha} f, \qquad \alpha \in \Gamma(T^*_{\mathcal{O}}\mathcal{E}^*), \quad f \in C^\infty(\mathcal{O}) = \Gamma(S^0\mathcal{E}); \\ \rho^1_\alpha \theta &= \{\alpha, \theta\}, \qquad \theta \in \Gamma(\mathcal{E}); \\ \rho^{p_1+p_2}_\alpha(s_1 \cdot s_2) &= (\rho^{p_1}_\alpha s_1) \cdot s_2 + s_1 \cdot \rho^{p_2}_\alpha s_2, \qquad s_i \in \Gamma(S^{p_i}\mathcal{E}) \end{aligned}$$

(here the dot stands for symmetric multiplication in $S^p\mathcal{E}$). Define the standard coboundary operator $D_p \colon C^k_p \to C^{k+1}_p$ in $C^k_p = \Gamma(\bigwedge^k T_{\mathcal{O}}\mathcal{E}^* \otimes S^p\mathcal{E})$ associated to the representation ρ^p_α:

$$\begin{aligned} D_p\eta(\alpha_0, \dots, \alpha_k) &= \sum_{j=0}^{k} (-1)^j \rho^p_{\alpha_j} \eta(\alpha_0, \dots, \widehat{\alpha_j}, \dots, \alpha_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} \eta(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_k), \quad \eta \in C^k_p. \end{aligned}$$

Denote by $\mathcal{H}^k_p$ the cohomology space of the operator D_p.

THEOREM 7.1. *Let the symplectic manifold $\mathcal{O}$ be compact. If two proper Poisson brackets on $\mathcal{E}^*$ induce the same transitive Lie algebroid's structure on $T^*_{\mathcal{O}}\mathcal{E}^*$ such that*

$$\mathcal{H}^2_p = 0 \qquad \text{for any} \quad p \geq 2, \tag{7.3}$$

then these Poisson brackets are formally equivalent.

To give a sketch of the proof, we need the following lemmas.

LEMMA 7.1. *Two proper Poisson brackets with Poisson tensors $\Psi, \Psi_1 \in \mathcal{V}^2(\mathcal{E}^*)$ define the same Lie algebroid structure in $T^*_{\mathcal{O}}\mathcal{E}^*$ if and only if $\Psi - \Psi_1 \in \mu^2\mathcal{V}^2(\mathcal{E}^*)$.*

Note that $\mu^p\mathcal{V}^k(\mathcal{E}^*)/\mu^{p+1}\mathcal{V}^k(\mathcal{E}^*) \simeq C^k_p$ and denote the natural projection by $\pi_p \colon \mu^p(\mathcal{E}^*) \to C^k_p$. Let $\Psi \in \mathcal{V}^2(\mathcal{E}^*)$ be the Poisson tensor of a proper Poisson bracket, and let $D_\Psi \colon \mathcal{V}^k(\mathcal{E}^*) \to \mathcal{V}^{k+1}(\mathcal{E}^*)$ be the corresponding Lichnerowicz operator (1.6).

LEMMA 7.2. *The following relations hold:*

$$[\![\mu^{p_1}\mathcal{V}^{k_1}(\mathcal{E}^*), \mu^{p_2}\mathcal{V}^{k_2}(\mathcal{E}^*)]\!] \subseteq \mu^{p_1+p_2-1}\mathcal{V}^{k_1+k_2-1}(\mathcal{E}^*);$$

$$D_\Psi \mu^p \mathcal{V}^k(\mathcal{E}^*) \subseteq \mu^p \mathcal{V}^{k+1}(\mathcal{E}^*); \tag{7.4}$$

$$\pi_p D_\Psi Q = D_p \pi_p Q \qquad \text{for all} \quad Q \in \mu^p\mathcal{V}^k(\mathcal{E}^*). \tag{7.5}$$

LEMMA 7.3. *Let $\mathcal{O}$ be compact and $X \in \mu^2\mathcal{V}^1(\mathcal{E}^*)$. Then there is a one-parameter family of local diffeomorphisms* $\exp(tX)$ *such that*

$$\exp(tX)\big|_{\mathcal{O}} = \mathrm{id}_{\mathcal{O}} \quad \textit{and} \quad \frac{d}{dt}\exp(tX)_*\Phi = \mathcal{L}_X \exp(tX)_*\Phi$$

for any tensor field Φ.

(The last lemma follows from more general facts in the theory of ordinary differential equations (see, for example, [**NS**]).)

The decomposition of $T^*_{\mathcal{O}}\mathcal{E}^*$ induced by the zero section gives a transversal subbundle (3.8), hence naturally defines an adjoint connection ∇ (3.10). Recall that a linear connection in $\mathcal{E} \to \mathcal{O}$ defines a linear connection in $\mathcal{E}^* \to \mathcal{O}$, hence a decomposition of $T\mathcal{E}^*$ into horizontal and vertical subbundles. This decomposition defines a mapping $\chi^{\nabla}_{p,k}\colon C^k_p \to \mu^p\mathcal{V}^k(\mathcal{E}^*)$ such that $\chi^{\nabla}_{0,1}\colon \Gamma(T_{\mathcal{O}}\mathcal{E}^*) \to \mu^1(\mathcal{E}^*)$ maps two summands of the decomposition $T_{\mathcal{O}}\mathcal{E}^* = T\mathcal{O} \oplus \mathcal{E}^*$ into the horizontal and vertical subbundles, respectively, and $\chi^{\nabla}_{p,k}$ satisfies the following properties:

$$\pi_p \circ \chi^{\nabla}_{p,k} = \mathrm{id}_{C^k_p},$$
$$\chi^{\nabla}_{(p_1+p_2),(k_1+k_2)}(\eta_1 \overline{\wedge} \eta_2) = \chi^{\nabla}_{p_1,k_1}\eta_1 \wedge \chi^{\nabla}_{p_2,k_2}\eta_2$$

(here $\eta_i \in C^{k_i}_{p_i}$, $i = 1, 2$, and $\overline{\wedge}\colon C^{k_1}_{p_1} \times C^{k_2}_{p_2} \to C^{k_1+k_2}_{p_1+p_2}$ is the exterior multiplication),

$$\chi^{\nabla}_{0,0} = p^* \qquad (\text{here} \quad p\colon \mathcal{E}^* \to \mathcal{O}),$$

and $\chi^{\nabla}_{1,0}$ provides the isomorphism between the sections of $\mathcal{E}$ and fiberwise linear functions of $\mathcal{E}^*$.

LEMMA 7.4. *Let $\Psi \in \mathcal{V}^2(\mathcal{E}^*)$ be a Poisson tensor of a proper Poisson bracket, and let $\Psi_1, \Psi_2 \in \mathcal{V}^2(\mathcal{E}^*)$ satisfy the conditions*

$$\sigma \overset{\text{def}}{=} \Psi_2 - \Psi_1 \in \mu^{p-1}\mathcal{V}^2(\mathcal{E}^*), \qquad p \geq 3, \tag{7.6}$$
$$\Psi_2 - \Psi \in \mu^2\mathcal{V}^2(\mathcal{E}^*). \tag{7.7}$$

Assume that a section $\eta \in C^1_{p-1}$ is a solution of the homological equation

$$D_{p-1}\eta = \pi_{p-1}\sigma. \tag{7.8}$$

Then the vector field

$$X \overset{\text{def}}{=} \chi^{\nabla}_{p-1,1}\eta \tag{7.9}$$

satisfies the conditions

$$\exp(tX)_*\Psi_2 - \Psi \in \mu^2\mathcal{V}^2(\mathcal{E}^*), \tag{7.10}$$
$$\frac{d}{dt}\big(\exp(tX)_*\Psi_2\big) + \sigma \in \mu^p\mathcal{V}^2(\mathcal{E}^*). \tag{7.11}$$

(*Here* $\exp(tX)$ *is a one-parameter group of local diffeomorphisms for the vector field* X.)

Using the above lemmas, we can give the sketch of the proof of Theorem 7.1.

PROOF. Let Ψ be a Poisson tensor of a proper Poisson bracket on $\mathcal{E}^*$ and let $\Psi_1 \in \mathcal{V}^2(\mathcal{E}^*)$ be such that $\Psi - \Psi_1 \in \mu^2\mathcal{V}^2(\mathcal{E}^*)$. To prove that for any $p \geq 2$ there exists a local diffeomorphism g such that its restriction to $\mathcal{O}$ is the identity and (7.2) holds, we shall use induction on p. If $p = 2$, then $g = \mathrm{id}$. Assume that there exists a local diffeomorphism g_1 such that $g_1\big|_{\mathcal{O}} = \mathrm{id}_{\mathcal{O}}$ and $g_{1*}\Psi - \Psi_1 \in \mu^{p-1}\mathcal{V}^2(\mathcal{E}^*)$.

Let us denote $\Psi_2 \overset{\text{def}}{=} g_{1*}\Psi$ and σ as in (7.6). Applying π_{p-1} to the equality $[\![\Psi_2, \Psi_2]\!] = 0$ and using formulas (7.6) and (7.7), we get $D_{p-1}\pi_{p-1}\sigma = 0$. Since $\mathcal{H}^2_{p-1} = 0$, the homological equation (7.8) has a solution; therefore, we can define a one-parameter group of local diffeomorphisms $\exp(tX)$ of the vector field (7.9).

Define a local diffeomorphism $g \overset{\text{def}}{=} \exp(tX)\big|_{t=1} \circ g_1$. Since $X \in \mu^{p-1}\mathcal{V}^1(\mathcal{E}^*) \subset \mu^2\mathcal{V}^1(\mathcal{E}^*)$, the restriction of g on $\mathcal{O}$ is identity. To prove that $g_*\Psi - \Psi_1 \in \mu^p\mathcal{V}^2(\mathcal{E}^*)$, we show that

$$\Phi_t = \exp(tX)_*\Psi_2 - (\Psi_2 - t\sigma) \in \mu^p\mathcal{V}^2(\mathcal{E}^*), \qquad \forall t \in \mathbb{R}. \tag{7.12}$$

Note that $\Phi_0 = 0$, and $\frac{d}{dt}\Phi_t = \frac{d}{dt}\exp(tX)_*\Psi_2 + \sigma \in \mu^p\mathcal{V}^2(\mathcal{E}^*)$ (by Lemma 7.3). Therefore, (7.2) holds. □

REMARK. The proof of Theorem 7.1 is an improved version of that for the case in which $\mathcal{O}$ is a point. A theorem similar to Theorem 7.1 for the case of a single point was proved by O. Lychagina [**Ly**].

Now we would like to discuss briefly some facts about the cohomology space $\mathcal{H}^k_p$ and to give a sufficient condition for vanishing of $\mathcal{H}^2_p$. All facts given in §4 can be easily generalized to the case of the cochain complex C^k_p and the operator D_p.

Consider the representation ρ^p of the Lie algebra $\Gamma(T^*_{\mathcal{O}}\mathcal{E}^*)$ in $\Gamma(S^p\mathcal{E})$. Let $(E^{s,t}_{(p)r}, d^{s,t}_{(p)r})$ denote the Hochshild–Serre spectral sequence [**HS**] related to the ideal $\Gamma(\mathcal{E})$ in the Lie algebra $\Gamma(T^*_{\mathcal{O}}\mathcal{E}^*)$.

THEOREM 7.2. *The spectral sequence $(E^{s,t}_{(p)r}, d^{s,t}_{(p)r})$ converges to the cohomology space $\mathcal{H}^k_p$, i.e.,*

$$\mathcal{H}^k_p \simeq \bigoplus_{s+t=k} E^{s,t}_{(p)r}, \qquad \text{for} \quad r \geq \max(t+1, s) + 1.$$

Moreover, if the base $\mathcal{O}$ is simply connected, then

$$E^{s,t}_{(p)2} \simeq H^t(\mathfrak{g}_x, S^p\mathfrak{g}_x) \otimes H^s(\mathcal{O}),$$

where $H^s(\mathcal{O})$ is the de Rham cohomology of the symplectic leaf $\mathcal{O}$ and $H^t(\mathfrak{g}_x, S^p\mathfrak{g}_x)$ is the cohomology of the finite-dimensional Lie algebra $\mathfrak{g}_x$ with coefficients in its pth symmetric product.

REMARK 7.3. Note that if the "curvature" (3.11a) of the adjoint connection in $\mathcal{E}$ induced by the decomposition of $T_{\mathcal{O}}\mathcal{E}^*$ vanishes ($R \equiv 0$), then the spectral sequence $(E^{s,t}_{(p)r}, d^{s,t}_{(p)r})$ degenerates in the second term, and given that $\mathcal{O}$ is simply connected, we have $\mathcal{H}^k_p = \bigoplus_{s+t=k} H^t(\mathfrak{g}_x, S^p\mathfrak{g}_x) \otimes H^s(\mathcal{O})$.

COROLLARY 7.4. *Let the symplectic leaf $\mathcal{O}$ be simply connected and*

$$H^2(\mathcal{O}) = 0; \tag{7.15}$$

also assume that the normal algebra generated in each fiber $\mathcal{E}_x = \mathfrak{g}_x$ by a proper Poisson bracket satisfies the condition

$$H^2(\mathfrak{g}_x, S^p\mathfrak{g}_x) = 0. \tag{7.16}$$

Then $\mathcal{H}_p^2 = 0$.

Finally, observe that condition (7.16) is satisfied for each $p \geq 0$ if the Lie algebra $\mathfrak{g}_x$ is isomorphic to the direct sum of its one-dimensional center and a semisimple Lie algebra. Together with (7.15) and the assumption that $\mathcal{O}$ is simply connected, this implies (7.3); however, in this case, the symplectic leaf cannot be compact. So, to ensure a formal equivalence of two proper Poisson brackets, the vector fields (7.9) must be complete.

Appendix. The de Rham cohomology of flat connections

In this section we collect well-known facts [**BT, GHV, Li**$_1$] about flat connections and cohomology of locally constant vector bundles, which we use in the present paper.

Let $\mathcal{E} \to B$ be a vector bundle. The operation ∇ that to each vector field $v \in \mathcal{V}^1(B)$ assigns a mapping of sections $\nabla_v : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ and satisfies the axioms

$$\nabla_{\varphi v_1 + v_2}\eta = \varphi\nabla_{v_1}\eta + \nabla_{v_2}\eta, \qquad \eta \in \Gamma(\mathcal{E}), \quad \varphi \in C^\infty(B), \tag{A.1}$$

$$\nabla_v(\varphi\eta_1 + \eta_2) = \varphi\nabla_v\eta_1 + \nabla_v\eta_2 + \mathcal{L}_v(\varphi)\eta_1, \qquad \eta_1, \eta_2 \in \Gamma(\mathcal{E}). \tag{A.2}$$

is called a *connection in the vector bundle.*

The connection ∇ in the vector bundle $\mathcal{E}$ naturally generates connections in the bundles $\mathcal{E}^*$, $\bigwedge^t \mathcal{E}$, and $\mathcal{E}/\mathcal{E}_1$, where $\mathcal{E}_1 \subset \mathcal{E}$ is a ∇-invariant subbundle.

A section $K \in \Gamma(\mathrm{Hom}(\mathcal{E}) \otimes \bigwedge^2 T^*B)$, defined by the formula

$$K(v_1, v_2)\eta \stackrel{\text{def}}{=} \nabla_{v_1}\nabla_{v_2}\eta - \nabla_{v_2}\nabla_{v_1}\eta - \nabla_{[v_1,v_2]}\eta, \quad v_1, v_2 \in \mathcal{V}^1(B), \;\; \eta \in \Gamma(\mathcal{E}), \tag{A.3}$$

is called the *curvature* of the connection ∇.

The connection ∇ is called *flat* if it has zero curvature. The parallel transport of a flat connection depends only on the homotopy class of the path [**Li**$_1$]; therefore, in any simply connected domain U there is a basis of parallel sections, obtained by the parallel transport of the basis in some fiber $\mathcal{E}_{x_0}$ ($x_0 \in U$), and the holonomy group of the connection ∇ is a homomorphism of the fundamental group $\pi_1(B)$ into the group $GL(\mathcal{E}_{x_0})$. In particular, if the base is simply connected, then the holonomy group of a flat connection is trivial and the bundle $\mathcal{E}$ is trivial ($\mathcal{E} = \mathcal{E}_{x_0} \times B$).

Let us define a space of cochains $C^k(\mathcal{E}) \stackrel{\text{def}}{=} \Gamma(\mathcal{E} \otimes \bigwedge^k T^*B)$ and an operator $\nabla : C^k(\mathcal{E}) \to C^{k+1}(\mathcal{E})$, which we denote by the same symbol as the connection,

$$\begin{aligned}\nabla\eta(v_0, v_1, \ldots, v_k) &\stackrel{\text{def}}{=} \sum_{j=0}^{k}(-1)^j\nabla_{v_j}\eta(v_0, v_1, \ldots, \widehat{v}_j, \ldots, v_k)\\ &\quad + \sum_{0\leq i<j\leq k}(-1)^{i+j}\eta([v_i, v_j], v_0, v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_k).\end{aligned} \tag{A.4}$$

It is easy to see that the operator ∇^2 has the form

$$\nabla^2\eta(v_0, v_1, \dots, v_k, v_{k+1}) = -\sum_{0\le i<j\le k+1} (-1)^{i+j} K(v_i, v_j)\eta(v_0, \dots, \widehat{v}_i, \dots, \widehat{v}_j, \dots, v_{k+1}).$$

Therefore, we have the following statement.

LEMMA A.1. *A connection is flat if and only if* $\nabla^2 = 0$.

For each flat connection ∇, by $Z^k_\nabla(\mathcal{E})$, $B^k_\nabla(\mathcal{E})$, and $H^k_\nabla(\mathcal{E}) = Z^k_\nabla/B^k_\nabla(\mathcal{E})$ we denote cocycles, coboundaries, and the cohomology of the operator (A.4). The cohomology space $H^{\cdot}_\nabla(\mathcal{E}) = \bigoplus_k H^k_\nabla(\mathcal{E})$ depends on the choice of the flat connection ∇^k. We have the following statement [**BT,** Prop. 7.4].

LEMMA A.2. *The connection* ∇ *has a trivial holonomy group if and only if*

$$H^k_\nabla(\mathcal{E}) \simeq H^k(B) \otimes \mathcal{E}_{x_0} \qquad \forall\, k \ge 0 \tag{A.5}$$

(here $H^k(B)$ *is the de Rham cohomology of the base).*

LEMMA A.3. *Suppose that the base* B *is a finite-type manifold. Then for any flat connection* ∇ *the space* $H^{\cdot}_\nabla(\mathcal{E})$ *is finite-dimensional.*

To prove Lemma A.3 we apply the Mayer–Vietoris sequence and repeat the proof of [**BT,** Prop. 5.3.1] for the fact that the de Rham cohomology is finite-dimensional.

In Section 4 we have used a slightly generalized version of Lemma A.2. Suppose that $(A \to B, q, \{\,,\})$ is a transitive Lie algebroid, and $\mathcal{E} \to B$ is a vector bundle. We define the space of cochains $C^k(A, \mathcal{E}) = \Gamma(\mathcal{E} \otimes \bigwedge^k A^*)$. For each connection ∇ in the bundle $\mathcal{E}$ we define the operator $\overline{\nabla}\colon C^k(A, \mathcal{E}) \to C^{k+1}(A, \mathcal{E})$ as follows:

$$\begin{aligned}\overline{\nabla}\eta(\alpha_0, \dots, \alpha_k) &\overset{\text{def}}{=} \sum_{j=0}^{k} (-1)^j \nabla_{q\alpha_j}\eta(\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_k) \\ &\quad + \sum_{0\le i<j\le k} (-1)^{i+j}\eta(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \widehat{\alpha}_i, \dots, \widehat{\alpha}_j, \dots, \alpha_k)\end{aligned} \tag{A.6}$$

(here $\eta \in C^k(A, \mathcal{E})$, $\alpha_j \in \Gamma(A)$). Obviously, the connection ∇ is flat if and only if $\overline{\nabla}^2 = 0$. By $H^k_\nabla(A, \mathcal{E})$ we denote the cohomology of the operator (A.6).

LEMMA A.4. *The holonomy group of the flat connection* ∇ *is trivial if and only if*

$$H^k_\nabla(A, \mathcal{E}) \simeq \mathcal{H}^k(A) \otimes \mathcal{E}_{x_0} \qquad \forall\, k \ge 0, \tag{A.7}$$

where $\mathcal{H}^k(A)$ *is the cohomology of a transitive Lie algebroid* A *and* $\mathcal{E}_{x_0}$ *is a fiber of the bundle* $\mathcal{E}$.

The proof of this lemma is similar to that of Lemma A.2.

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