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ORBIT REDUCTION OF CONTACT IDEALS

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ABSTRACT. For a given PDE system possessing a Lie group of internal symmetries the orbit reduction procedure is introduced and studied. The solutions of the reduced system are in one-to-one correspondence with the moduli space of regular solutions of the original system. The isomorphism between the local characteristic cohomology of the reduced unconstrained jet space and the Lie algebra cohomology of the symmetry group is established. The group-invariant Euler-Lagrange equations of an invariant variational problem are described as a composition of the Euler-Lagrange operators on the reduced jet space and certain other differential operators on the reduced jet space.

1. INTRODUCTION.

In this paper we introduce the orbit reduction of partial differential equations or, more generally, exterior differential systems. Recall, that an exterior differential system [4] is a pair $(M, \mathcal{I})$ where M is a manifold, and $\mathcal{I} \subset \bigwedge T^*M$ is a graded differentially closed ideal. It is a geometrical generalization of partial differential equations (in this case M is a submanifold of the jet space and $\mathcal{I}$ is the contact ideal).

Let $\mathcal{E} = (\Delta, \mathcal{I})$ be a system of partial differential equations, or more generally, an exterior differential system, invariant under the action of a finite-dimensional Lie group G of internal symmetries. The action of G on $\mathcal{E}$ induces a G-action on the space $\mathrm{Sol}(\mathcal{E})$ of the solutions of $\mathcal{E}$. Let $\mathcal{E}^{(r)} = (\Delta^{(r)}, \mathcal{I}^{(r)})$ be the r-th order prolongation of $\mathcal{E}$. The orbit space $\Delta^{(r)}/G$ possesses the structure of an exterior differential system induced by the structure of $\mathcal{E}^{(r)}$.

It turns out (see Theorem 2) that for high enough order r of prolongation the solutions of the reduced system are in one-to-one correspondence with the moduli space $\mathrm{Sol}(\mathcal{E})/G$ of almost all solutions of the original system. This motivates the studying of a group-invariant PDE system through the study of its reduced exterior differential system.

The other important reason for studying the orbit reduction is the *inverse problem of reduction.* By inverse reduction we mean the following. Given a certain system of nonlinear PDEs one may ask a question whether it is an orbit reduction of a different system of PDEs that has a simpler structure. The questions about the solutions of the original system translate into questions about the solutions of the "simpler" system. For example it would be interesting to identify the class of PDEs which are the orbit reduction of an unconstrained jet space. In this case knowing the inverse reduction gives the general solution of the original equations.

As the very first step towards the understanding the inverse reduction, we establish the isomorphism between the local characteristic cohomology of the reduced

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jet space and the Lie algebra cohomology of a Lie group of contact transformations acting on the jet space (see Theorem 3 in this paper). This in particular, implies that in order to realize a PDE system having an infinite-dimensional characteristic cohomology as an orbit reduction of a jet space one needs to consider actions of infinite-dimensional groups.

The other purpose of the present paper is to understand the group-invariant variational problems via the orbit reduction. As first observed by Sophus Lie [8], the Euler-Lagrange equations of every invariant variational problem can be written in terms of the differential invariants of the group action. In other words, the Euler-Lagrange equations of a group-invariant variational problem can be pushed forward to the orbit space. Surprisingly, up to date there was no general understanding of the meaning of the pushed forward equations on the orbit space, nor there was a general algorithm of producing the group-invariant Euler-Lagrange equations.

The reduced jet space has its own calculus of variations (for example Euler-Lagrange operators), that can be interpreted as a calculus of variations with constrains imposed by the syzygies of the differential invariants. It is well-understood that all the basic ingredients of such calculus of variations come from the edge complex of the corresponding Vinogradov spectral sequence [13]. We show (see Theorem 4 below) that for every invariant variational problem the push-forward of the invariant Euler-Lagrange equations onto the orbit space is a composition of the Euler-Lagrange operators on the reduced jet space and certain other differential operators. These other differential operators come from the morphism of the two Vinogradov spectral sequences of the original and the reduced jet spaces. Here we would like to note that an alternative approach for computing invariant Euler-Lagrange equations was recently proposed by I. Kogan and P. Olver ; see [6] and their contribution to these proceedings.

The proofs of all the results given in this paper can be found in [5].

2. Preliminaries: EDS and PDEs.

All the geometrical objects considered in this paper are of class C^∞ unless stated otherwise. All the considered manifolds are paracompact.

Let $\mathcal{I} = \cup_{x\in M}\mathcal{I}_x$ be a collection of homogeneous[1] ideals

$$\mathcal{I}_x = \oplus_{n=1}^{\dim(M)}\mathcal{I}_x^n \subset \bigwedge T_x^*M$$

in the graded exterior algebra $\bigwedge T_x^*M$. We shall say that a differential form $\omega \in \Omega^n(M)$ is a section of $\mathcal{I}$ ($\omega \in \Gamma(\mathcal{I})$) if for every $x \in M$, $\omega(x) \in \mathcal{I}_x^n$. The sections of $\mathcal{I}$ form a *differential ideal* if $d\Gamma(\mathcal{I}) \subset \Gamma(\mathcal{I})$. We shall assume that $\Gamma(\mathcal{I})$ does not contain any functions except zero.

Definition 2.1. *We shall say that $\mathcal{E} = (M,\mathcal{I})$ is an Exterior Differential System (or EDS for short) if the space of sections of $\mathcal{I}$ is a differential ideal, and there exists a closed subset $X_{\text{nonfree}} \subset M$ of zero measure, such that for every connected component $U \subset (M \setminus X)$ $\mathcal{I}_{|U} = \cup_{x\in U}\mathcal{I}_x$ is a subbundle of $\bigwedge T^*U$.*

In practice it is convenient to define $\mathcal{I}$ by the generators of $\Gamma(\mathcal{I})$. We shall say that $\Gamma(\mathcal{I})$ is generated by the forms $\omega_1, .., \omega_N$ (the notation is $\mathcal{I} =< \omega_1, .., \omega_n >$) if

[1] By saying that the ideal $\mathcal{I}_x$ is homogeneous we mean that in the homogeneous-degree decomposition $\omega_x = \omega_x^1 + .. + \omega_x^{\dim(M)}$ of $\omega \in \mathcal{I}_x$ every homogeneous element $\omega_x^n \in \bigwedge^n T^*M$ belongs to the ideal.

for every $\omega \in \Gamma(\mathcal{I})$ there exist forms $\alpha_i, \beta_i \in \Omega(M)$ such that $\omega = \sum_{i=1}^{N}(\omega_i \wedge \alpha_i + d\omega_i \wedge \beta_i)$. An EDS $\mathcal{E}$ is called *Pfaffian* if its ideal $\mathcal{I}$ is generated by 1-forms.

Definition 2.2. *A k-dimensional solution $S \in \mathrm{Sol}_k(\mathcal{E})$ of $\mathcal{E} = (M, \mathcal{I})$ is a connected k-dimensional submanifold $S \hookrightarrow M$, such that the pullback of $\Gamma(\mathcal{I})$ to S is zero.*

Example 2.1. Jet spaces. Let N be a manifold. Consider the r-th order jet space $J^r_k N \to N$ of k-dimensional submanifolds together with the standard contact ideal $\mathcal{C}^{(r)} \subset \bigwedge T^* J^r_k N$ (see for example [10]). For every k-dimensional submanifold $S \overset{i_S}{\hookrightarrow} N$ there is a natural lift $j^r i_S : S \hookrightarrow J^r_k N$ such that $\theta \in \Gamma(\mathcal{C}^{(r)})$ if and only if the pullback of θ by $j^r i_S$ is zero for every k-dimensional submanifold S. The lifts $j^r S = j^r i_S(S)$ are the solutions of the EDS $(J^r_k N, \mathcal{C}^{(r)})$.

Example 2.2. PDE systems. Let $\Delta \overset{\iota}{\hookrightarrow} J^r_k N$ be a subbundle of the jet space $J^r_k N$. This subbundle can be thought of as a system of partial differential equations, whose solutions are k-dimensional submanifolds $S \hookrightarrow N$ such that $j^r S \subset \Delta$. The lifts $j^r S$ of the solutions of Δ are the solutions of the EDS $\mathcal{E} = (\Delta, \iota^* \mathcal{C}^{(r)})$.

Note that since the contact ideals on the jet spaces are always generated by one-forms, not every EDS is described by the last example. However the prolongation [4] of every EDS is a first-order PDE system.

Recall that a (k-dimensional) prolongation [4] of $\mathcal{E} = (M, \mathcal{I})$ is an EDS $\mathcal{E}^{(1)}_k = (M^{(1)}_k, \mathcal{I}^{(1)}_k)$, where $M^{(1)}_k$ is a set of all k-dimensional planes in TM annihilating the ideal $\mathcal{I}$:

$$M^{(1)}_k = \{(P, z) |\; z \in M,\; P \subset T_z M,\; \dim(P) = k, \text{ and } \; \mathcal{I}_{z|P} = 0\} \overset{\iota_1}{\hookrightarrow} J^1_k M,$$

$$\mathcal{I}^{(1)}_k = \iota_1^* \mathcal{C}^{(1)}.$$

We shall always assume that $\pi^1 : M^{(1)}_k \to M$ is a smooth fiber bundle. Sometimes it will mean that we remove some closed subset from $M^{(1)}_k$ to make it smooth.

For every k-dimensional solution $S \hookrightarrow M$ its lift $j^1 S \hookrightarrow J^1_k M$ is a submanifold of $M^{(1)}_k$ and is a solution of the prolonged EDS $\mathcal{E}^{(1)}_k$. Conversely, given a solution $S_1 \hookrightarrow M^{(1)}_k$ of the prolonged EDS the natural projection $\pi^1(S_1) \subset M$ is a solution of the original EDS. However this projection may "lose" some of its dimension and may happen not to be a smooth manifold anymore.

Definition 2.3. *We shall say that an EDS $\mathcal{E}$ is of infinite type if for every $r > 1$ $M^{(r)}_k \to M^{(r-1)}_k$ is a differentiable fiber bundle, and* $\dim M^{(r)}_k - \dim M^{(r-1)}_k > 0$.

Example 2.3. Prolongation of PDE systems. Consider the Example 2.2. Denote by $\pi^r : J^r_k N \to N$ the natural projection. For each small enough open neighborhood $U \subset J^r_k N$ we may introduce local coordinates $x^1, .., x^k, u^1, .., u^q$ in $\pi^r(U) \simeq \mathbb{R}^{k+q}$ (this actually means that we artificially impose a structure of a fiber bundle $\pi^r(U) \to \mathbb{R}^k$). This choice of the coordinates on the base N induces the canonical jet coordinates (see for example [10, 2]) $(x^i, u^\alpha, u^\alpha_I)$ (here $I = (I_1, .., I_l)$ is a multi-index of length $l = |I| \leq r$). The contact ideal $\mathcal{C}^{(r)}$ is generated by the following 1-forms:

$$\{\theta^\alpha_J = du^\alpha_J - u^\alpha_{Ji} dx^i\}_{|J|<r} \tag{1}$$

Any subbundle $\Delta \hookrightarrow J^r_k N$ can be represented as a zero level set of functions $\Delta_\nu \in C^\infty(J^r_k N)$. Denote by $\frac{d}{dx^i} : C^\infty(J^r_k N) \to C^\infty(J^{r+1}_k N)$ the total derivatives w.r.t.

x^i. The PDE system

$$\Delta^{(1)} \stackrel{\text{def}}{=} \{ \Delta_\nu = 0, \ \frac{d}{dx^i}\Delta_\nu = 0 \} \stackrel{\iota'}{\hookrightarrow} J_k^{r+1}N$$

is called a prolongation of Δ (see for example [7, 9]).

The prolongation of an EDS can be iterated thus giving a prolongation tower

$$M \leftarrow M_k^{(1)} \leftarrow M_k^{(2)} \leftarrow \cdots \leftarrow M_k^{(\infty)},$$

where $M_k^{(\infty)} = \lim_{r\to\infty} M_k^{(r)}$ is the inverse limit.

3. The reduced EDS.

Let G be some pseudogroup of local diffeomorphisms acting on a manifold M. We shall say that an EDS $\mathcal{E} = (M, \mathcal{I})$ is G-invariant if for every $x \in M$ and $g \in G$

$$g^*\mathcal{I}_{gx} = \mathcal{I}_x \tag{2}$$

If $\mathcal{E} = (\Delta, \iota^*\mathcal{C}^{(r)})$ as in Example 2.2 then the symmetry group G is usually called a group of internal symmetries [14, 3].

We shall always assume that the orbit space $\overline{M} \stackrel{\text{def}}{=} M/G$ is again a differentiable manifold (in what follows we shall always denote the orbit spaces by barred symbols). The local coordinates on $\bar{M}$ may be identified with the G-invariant functions on M. The local coordinates on $\overline{M_k^{(r)}}$ are usually called the differential invariants of order r of the G-action.

Proposition 3.1. *Let $\mathcal{E} = (M, \mathcal{I})$ be a G-invariant exterior differential system, then there exists an exterior differential system $\overline{\mathcal{E}} = (\overline{M}, \overline{\mathcal{I}})$, such that $\overline{\mathcal{I}}$ is the maximal* [2] *ideal satisfying*

$$\mathfrak{p}^*\overline{\mathcal{I}}_{\mathfrak{p}(x)} \subset \mathcal{I}_x \qquad \forall x \in M, \tag{3}$$

where $\mathfrak{p} : M \to \overline{M} = M/G$ is the natural projection.

Definition 3.2. *We shall call $\overline{\mathcal{E}} = (\overline{M}, \overline{\mathcal{I}})$ the reduced EDS.*

Example 3.1. Consider the action of the abelian group $G = \mathbb{R}^3$ on itself ($M = \mathbb{R}^3$) by translations. Define $\mathcal{E} = (\mathbb{R}^3, < 0 >)$. A group action on M can be always lifted to a group action on the prolonged manifold $M_k^{(r)}$. The two-dimensional ($k = 2$) r-th prolongation of $\mathcal{E}$ is the jet space of two-dimensional submanifolds:

$$\mathcal{E}_2^{(r)} = (J_2^r\mathbb{R}^3, \mathcal{C}^{(r)}).$$

In order to coordinatize the orbit spaces $\overline{J_2^r\mathbb{R}^3}$, we introduce the coordinates (x^1, x^2, u) in $\mathbb{R}^3$ as well as the standard jet coordinates u_I in the fibers of $J_2^r\mathbb{R}^3$ (here I is a multi-index). Note that in fact we restricted our attention to the coordinate chart $U_r \subset J_2^r\mathbb{R}^3$ that has a complement of zero Borel measure in $J_2^r\mathbb{R}^3$. The orbit space $\overline{U}_r = U_r/\mathbb{R}^3$ is a Euclidean space with coordinates u_I, where $1 \leq |I| \leq r$.

Denote by

$$y^i \stackrel{\text{def}}{=} u_i, \quad i = 1, 2 \tag{4}$$

[2] By saying that $\overline{\mathcal{I}}$ is maximal we mean that any other ideal that satisfies condition (3) is contained in $\overline{\mathcal{I}}$.

the coordinates on the orbit space $\overline{J_2^{(1)}\mathbb{R}^3} \simeq RP^2$. It is obvious that the reduced ideal $\overline{\mathcal{C}^{(1)}}$ is trivial, thus

$$\overline{\mathcal{E}_2^{(1)}} = (RP^2, < 0 >).$$

The contact ideal on $J_2^2\mathbb{R}^3$ is generated by the three $\mathbb{R}^3$-invariant 1-forms

$$\eta^1 = du - u_1 dx^1 - u_2 dx^2, \tag{5}$$

$$\eta^2 = du_1 - u_{11} dx^1 - u_{12} dx^2, \tag{6}$$

$$\eta^3 = du_2 - u_{12} dx^1 - u_{22} dx^2. \tag{7}$$

Let us introduce the coordinates on the fiber of $\overline{J_2^2\mathbb{R}^3} \to \overline{J_2^1\mathbb{R}^3}$:

$$v^1 = u_{11},\ v^2 = u_{22},\ v^3 = u_{12}. \tag{8}$$

Direct calculations show that the reduced ideal $\overline{\mathcal{C}^{(2)}} \subset \bigwedge T^*\overline{J_2^2\mathbb{R}^3}$ has no 1-form component, however it does have a nontrivial 2-form component, generated by the 2-forms $\bar{\omega}_1, \bar{\omega}_2 \in \Omega^2(\overline{J_2^2\mathbb{R}^3})$,

$$\bar{\omega}_1 = (v^2 dy^1 - v^3 dy^2) \wedge dv^1 + (v^1 dy^2 - v^3 dy^1) \wedge dv^3 =$$
$$= (u_{11}u_{22} - u_{12}^2) d\eta^2 + (u_{22}\eta^2 - u_{12}\eta^3) \wedge du_{11} + (u_{11}\eta^3 - u_{12}\eta^2) \wedge du_{12},$$

$$\bar{\omega}_2 = (v^2 dy^1 - v^3 dy^2) \wedge dv^3 + (v^1 dy^2 - v^3 dy^1) \wedge dv^2 =$$
$$= (u_{11}u_{22} - u_{12}^2) d\eta^3 + (u_{22}\eta^2 - u_{12}\eta^3) \wedge du_{12} + (u_{11}\eta^3 - u_{12}\eta^2) \wedge du_{22}$$

(in fact $\overline{\mathcal{C}^{(2)}}$ is generated by its 2-form component). Therefore

$$\overline{\mathcal{E}_2^{(2)}} \simeq (RP^2 \times \mathbb{R}^3, < \bar{\omega}_1, \bar{\omega}_2 >).$$

This example shows that although the original EDS is generated by 1-forms, the reduced EDS does not necessarily have the same property. In particular, it may not be a prolongation of anything. This raises the natural question of whether the reduction procedure commutes with the prolongation. We address this question in Theorem 1 below.

4. Syzygies of Differential Invariants.

Consider a Lie group G, acting on M and G-invariant EDS $\mathcal{E} = (M, \mathcal{I})$. The action of G on M prolongs to the action on $M_k^{(r)}$. It is well-known [11, 10] that if $\mathcal{E} = (M, \mathcal{I})$ is an infinite-type EDS and the action is effective on open subsets then the G-action is locally free (i.e. the stabilizers are discrete) almost everywhere on $M_k^{(r)}$ for big enough r. The author is not aware of any example when the action does not eventually become free on high enough prolongation. Moreover, in the real-analytic category there are strong indications that every effective action becomes free on high enough prolongation [1]. Throughout this paper we shall adopt the following hypothesis:

The Main Assumptions.

1. G is a finite-dimensional Lie group and the given EDS $\mathcal{E}$ is of infinite type.
2. There exists an integer r_s and a closed subset $X_{\text{nonfree}} \subset M_k^{(r_s)}$ of zero Borel measure such that the action of G is free on $M_k^{(r_s)} \setminus X_{\text{nonfree}}$.
3. The quotient space $\overline{M_k^{(r_s)}} = (M_k^{(r_s)} \setminus X_{\text{nonfree}})/G$ is a differentiable manifold.
4. There exists an integer r_{cf}, and a closed subset $X_{\text{nontr}} \subset M_k^{(r_{\text{cf}}+1)}$ such that every k – *dimensional* integral element P of the EDS $\mathcal{E}_k^{(r_{\text{cf}})}$ that lies outside of the set X_{nontr} (i.e. $P \in M_k^{(r_{\text{cf}}+1)} \setminus X_{\text{nontr}}$) is transversal to the orbits of the G-action.

The last condition means that almost all the solutions of the considered EDS are "maximally noninvariant" in the classification of partially invariant solutions due to Ovsiannikov [11]. This condition is automatically satisfied for every finite-dimensional Lie group action on trivial EDS $(M, < 0 >)$ (note that its prolongation is the unconstrained jet space). It is also satisfied for a very wide class of PDE systems. However there exist examples of EDS whose solutions are always nontransversal to the orbits of the action of its automorphism group. For example if we consider the EDS $(\mathbb{R}^3, < dx^1 \wedge dx^2 >)$ together with the action of $\mathbb{R}^3$ on itself, then every two-dimensional solution of this EDS is not transversal to the orbits.

For every $r_1 > r_2$ we denote by $\pi^{r_1}_{r_2} : M_k^{(r_1)} \to M_k^{(r_2)}$ the natural projection, and by $X^r \subset M_k^{(r)}$ we denote the subset where either transversality or freeness assumption fails:

$$X^r \overset{\text{def}}{=} (\pi^r_{r_s})^{-1}(X_{\text{nonfree}}) \cup (\pi^r_{r_{\text{cf}}})^{-1}(X_{\text{nontr}}).$$

Theorem 1. *Let the main assumptions hold, then for every $r \geq \max(r_s, r_{\text{cf}})+1$ the procedure of reduction of $\mathcal{E}_k^{(r)}$ commutes with the procedure of prolongation outside of the singular set X^r, i.e.*

$$\overline{(\dot{\mathcal{E}}_k^{(r)})_k^{(1)}} = \overline{\dot{\mathcal{E}}_k^{(r+1)}}, \qquad \textit{where} \quad \dot{\mathcal{E}}_k^{(r)} \overset{\text{def}}{=} (M_k^{(r)} \setminus X^r, \mathcal{I}_k^{(r)}).$$

Now we would like to reinterpret Theorem 1 in terms of the local coordinates. Denote by $\mathcal{E}_k^{(\infty)} = (M_k^{(\infty)}, \mathcal{I}_k^{(\infty)})$ the infinite prolongation of $\mathcal{E}$ and by $\mathfrak{p} : M_k^{(r)} \to \overline{M_k^{(r)}}$ denote the natural projection onto the orbit space.

Lemma 4.1. *Under the main assumptions* **1-3** *the transversality assumption* **4** *is equivalent to the following. There exists an integer $r_{\text{cf}} \geq 0$ and a closed subset $X_{\text{nontr}} \subset M_k^{(r_{\text{cf}}+1)}$ of zero measure, such that every point of $M_k^{(r_{\text{cf}}+1)} \setminus X_{\text{nontr}}$ has an open neighborhood $U \subset M_k^{(r_{\text{cf}}+1)} \setminus X_{\text{nontr}}$ and differential invariants of the G-action $y^1, .., y^k \in C^\infty(\mathfrak{p}(\pi^{r_{\text{cf}}+1}_{r_{\text{cf}}}(U)))$ that satisfy*

$$\mathfrak{p}^*(dy^1 \wedge .. \wedge dy^k) \notin \Gamma(\mathcal{I}_k^{(r_{\text{cf}}+1)}) \cap \Omega^k(U). \tag{9}$$

Choosing the differential invariants y^i allows us to introduce total differential operators $\frac{d}{dy^i} : C^\infty(\overline{M_k^{(r)}}) \to C^\infty(\overline{M_k^{(r+1)}})$ (here $r \geq r_{\text{cf}}$) in the following way. Denote by $[]_0 : \Omega^1(M_k^{(r)}) \to \Omega^1(M_k^{(r+1)})/\Gamma(\mathcal{I}_k^{(r+1)})$ the composition of $(\pi^{r+1}_r)^*$ and the natural projection to the quotient. The formula (9) implies that the forms

$[\mathfrak{p}^* dy^i]_0$ form a basis in this quotient. Then these operators are defined by the equality

$$[\mathfrak{p}^* dF]_0 = \sum_{i=1}^{k} (\mathfrak{p}^* \frac{dF}{dy^i})[\mathfrak{p}^* dy^i]_0 , \quad F \in C^\infty(\overline{M_k^{(r)}}).$$

Note also that the operators $\frac{d}{dy^i}$ commute with each other.

The following lemma originally appeared in the work of A. Tresse [12] in the context of the unconstrained jet spaces. It says that the differential invariants of any order are generated by taking total derivatives of finitely many differential invariants.

Lemma 4.2. *For every* $r \geq \max(r_s, r_{\rm cf}) + 2$ *the differential invariants of order* r *are obtained by taking the total derivatives of the invariants of order* $r-1$*:*

$$\forall f \in C^\infty(\overline{M_k^{(r)}}) \qquad f = f(y^i, v^a, \frac{dv^a}{dy^i})$$

where (y^i, v^a) *are local coordinates on* $\overline{M_k^{(r-1)}}$, $i = 1, .., k$, $a = 1, .., \tilde{q}$.

Remark 4.3. Generally speaking the number of "dependent variables" v^a necessary for generating of all the other differential invariants may be less than $\tilde{q} = \dim \overline{M_k^{(r_o - 1)}} - k$.

We may think of $(y^i, v^a, v^a_i = \frac{dv^a}{dy^i})$ as standard jet coordinates on $J^1_k \overline{M_k^{(r-1)}}$. These coordinate functions give the mapping $\iota_1 : \overline{M_k^{(r)}} \to J^1_k \overline{M_k^{(r-1)}}$. The image $\bar{\Delta}$ of ι_1 is a PDE system that can be described locally as a zero locus of functions $\overline{\Delta}_\nu \in C^\infty(J^1_k \overline{M_k^{(r-1)}})$. These functions are sometimes called *syzygies of differential invariants* [10].

Now the Theorem 1 implies that *for every* $r \geq r_o = \max(r_s, r_{\rm cf}) + 2$ *the reduced EDS* $\overline{\mathcal{E}_k^{(r)}}$ *is the* $(r - r_o)$*-th order prolongation of the syzygy PDE system* $\bar{\Delta} \hookrightarrow J^1_k \overline{M_k^{(r_o - 1)}}$.

Example 4.1. Consider the example 3.1. Here $r_s = 0$, and $r_{\rm cf} = 1$. On the space $\overline{J^2_2 \mathbb{R}^3}$ we introduced the local coordinates $(y^1, y^2, v^1, v^2, v^3)$. Since $r_o = 3$, all the higher order differential invariants are generated by the total derivatives of v^a. Counting the dimensions shows that there are two functionally independent syzygies, namely

$$\bar{\Delta}_1 = v^3(v^2_2 - v^3_1) + v^1 v^2_1 - v^2 v^3_2 = 0, \qquad \bar{\Delta}_2 = v^3(v^1_1 - v^3_2) + v^2 v^1_2 - v^1 v^3_1 = 0,$$

(here $v^a_i = \frac{dv^a}{dy^i}$).

5. Moduli space of solutions and the reduced EDS

Let G be a Lie group acting on M. Let $\mathcal{E} = (M, \mathcal{I})$ be a G-invariant EDS of infinite type. Denote by $\mathrm{Sol}_k(\mathcal{E})$ the space of k-dimensional solutions of $\mathcal{E}$. We shall say that a solution $S_r \in \mathrm{Sol}_k(\mathcal{E}_k^{(r)})$ is *regular* (the notation is $S_r \in \mathrm{Sol}_k^{\rm reg}(\mathcal{E}_k^{(r)}, G)$) if S_r is transversal to the orbits of the G-action on $M_k^{(r)}$ (clearly then the lifts of S_r to the higher prolongations are also regular).

For every $r > 0$ and every solution $S_r \in \mathrm{Sol}_k(\mathcal{E}_k^{(r)})$ we may consider the projection $\bar{S}_r = \mathfrak{p}(S_r) \subset \overline{M_k^{(r)}}$ of S onto the orbit space. If the solution S_r is regular then

$\bar{S}_r$ is a k-dimensional submanifold and is a solution of the reduced EDS $\overline{\mathcal{E}_k^{(r)}}$. It turns out that on "high enough" prolongation we can also lift a solution of $\overline{\mathcal{E}_k^{(r)}}$ to a dim G-parametric family of regular solutions of $\mathcal{E}_k^{(r)}$.

Theorem 2. *For every $r \geq \max(r_s, r_{\mathrm{cf}}) + 1$ the moduli space of regular solutions of the prolonged EDS is isomorphic to the solutions of the reduced EDS:*

$$\frac{\mathrm{Sol}_k^{\mathrm{reg}}(\mathcal{E}_k^{(r)}, G)}{G} \simeq \mathrm{Sol}_k \overline{\mathcal{E}_k^{(r)}} .$$

The procedure of lifting a k – *dimensional* solution $\bar{S} \hookrightarrow \overline{M_k^{(r)}}$ of the reduced EDS $\overline{\mathcal{E}_k^{(r)}}$ can be described as the following. Consider $\mathfrak{p}^{-1}(\bar{S}) \overset{i}{\hookrightarrow} M_k^{(r)}$. Define a differential ideal $\mathcal{J}(\bar{S}) \subset \bigwedge T^*\mathfrak{p}^{-1}(\bar{S})$ on the manifold $\mathfrak{p}^{-1}(\bar{S})$ by pulling back the ideal on $M_k^{(r)}$:

$$\mathcal{J}(\bar{S}) \overset{\mathrm{def}}{=} i^* \mathcal{I}_k^{(r)}.$$

Recall that an EDS $(M, \mathcal{J})$ is called Frobenius if it is algebraically generated by its 1-form component. In this case the manifold M is foliated by solutions of $(M, \mathcal{J})$.

Proposition 5.1. *Let $r \geq \max(r_s, r_{\mathrm{cf}}) + 1$, then the exterior differential system $(\mathfrak{p}^{-1}(\bar{S}), \mathcal{J}(\bar{S}))$ is Frobenius. The solutions of this EDS are transversal to the orbits of the G-action and form a foliation of codimension* $\dim G$.

Clearly, the solutions of this Frobenius EDS are the desired solutions of $\mathcal{E}_k^{(r)}$. Thus in practical terms the reconstruction of a solution of the original EDS from the solution of a reduced one consists of solving a sequence of k systems of ODEs.

6. Characteristic cohomology of the reduced jet spaces.

Let a Lie group G act on a manifold M. Suppose that the main assumptions **1-3** hold with regard to the trivial EDS $\mathcal{E} = (M, < 0 >)$. By virtue of Theorem 1 we may regard $(\overline{J_k^{(\infty)} M}, \overline{\mathcal{C}^{(\infty)}})$ as the infinite prolongation of the reduced EDS $\bar{\mathcal{E}}_0 = (\overline{J_k^{(\max(r_s, r_{\mathrm{cf}})+1)} M}, \overline{\mathcal{C}^{(\max(r_s, r_{\mathrm{cf}})+1)}})$ (or, equivalently, the infinite prolongation of a syzygy PDE system).

The fact that $\bar{\mathcal{E}}_0$ is a reduction of an unconstrained jet space allows us to deduce everything about the solutions of $\bar{\mathcal{E}}_0$, since every solution of $\bar{\mathcal{E}}_0$ is an image of a solution of $(J_k^{r_o} M, \mathcal{C}^{(r_o)})$ under the mapping $\mathfrak{p} : J_k^{r_o} M \to \overline{J_k^{(r_o)} M}$. Therefore it is important to investigate the conditions under which a given EDS $\bar{\mathcal{E}}_0$ can be a reduction of an unconstrained jet space.

It turns out that the local characteristic cohomology of the reduced EDS $\bar{\mathcal{E}}_0$ is isomorphic to the Lie algebra cohomology of the Lie group G.

Denote by $\bar{\pi}_r^\infty : \overline{J_k^\infty M} \to \overline{J_k^r M}$ the natural projection, and $r_1 = \max(r_s, r_{\mathrm{cf}})$.

Theorem 3. *For every open subset $\hat{U} \subset \overline{J_k^\infty M} \smallsetminus \mathfrak{p} \circ (\pi_{r_1}^\infty)^{-1}(X^{r_1})$, such that $\bar{\pi}_r^\infty \hat{U}$ is contractible for every $r \geq r_o$, the characteristic cohomology of $\bar{\mathcal{E}}_o$ over $\hat{U}$ is isomorphic to the Lie algebra cohomology of G in dimensions less than k:*

$$H^t(\Omega_{\mathrm{hor}}(\hat{U}), \bar{d}_0) \simeq H^t(\mathfrak{g}) \qquad \forall t < k, \tag{10}$$

where $\Omega_{\mathrm{hor}}^t(\hat{U}) \overset{\mathrm{def.}}{=} \Omega^t(\hat{U}) / \Gamma(\overline{\mathcal{C}^{(\infty)}})$, $\bar{d}_0$ is the horizontal differential induced on the horizontal forms $\Omega_{\mathrm{hor}}^t(\hat{U})$, and $H^t(\mathfrak{g})$ is the Lie algebra cohomology of the Lie group G.

7. Invariant variational problems.

Consider an unconstrained infinite jet space $J^\infty_k M$ of k-dimensional submanifolds of a manifold M. Denote by $(E^{s,t}_r, d^{s,t}_r)$ the Vinogradov spectral sequence [13] corresponding to the decreasing filtration

$$\mathcal{F}_s\Omega(J^\infty_k M) = \Omega(J^\infty_k M) \cap \wedge^s\Gamma(\mathcal{C}^{(\infty)}).$$

It is well-known [13, 2] that the k-dimensional variational problems on M can be identified with the space

$$E^{0,k}_1 = \Omega^k(J^\infty_k M)/\left(\Gamma(\mathcal{C}^{(\infty)}) + d\Omega^{k-1}(J^\infty_k M)\right),$$

and the Euler -Lagrange operator is $d^{0,k}_1 : E^{0,k}_1 \to E^{1,k}_1$, where the quotient

$$E^{1,k}_1 = \Gamma(\mathcal{C}^{(\infty)}) \cap \Omega^{k+1}(J^\infty_k M)/\left(\wedge^2\Gamma(\mathcal{C}^{(\infty)}) + d\Gamma(\mathcal{C}^{(\infty)})\right)$$

has a structure of a free module over the ring of functions on the infinite jet $J^\infty_k M$.

For a given $\lambda \in \Omega^k(J^r_k M)$ one may consider the Euler-Lagrange system $\mathrm{EL}(\lambda) \hookrightarrow J^{2r}_k M$ defined as the zero locus of $d^{0,k}_1[\lambda]^{0,k}_1$ (we denote by $[\lambda]^{0,k}_1$ the equivalence class in $E^{0,k}_1$). If $(x^i, u^\alpha, u^\alpha_J)$ are the standard jet coordinates in some open neighborhood of $J^\infty_k M$, $dx = dx^1\wedge\cdots\wedge dx^k$, and $\lambda = Ldx + \Gamma(\mathcal{C}^{(\infty)}) + d\Omega^{k-1}(J^\infty_k M)$ is the variational problem then the Euler-Lagrange system has the form

$$\mathrm{EL}(\lambda) = \{E_\alpha(L) = 0, \quad \alpha = 1,..,q = \dim M - k\}, \quad \text{where} \tag{11}$$

$$E_\alpha(L) = \sum_{|I|=0}^{r} (-1)^{|I|}\frac{d^{|I|}}{dx^I}\frac{\partial}{\partial u^\alpha_I}L,$$

$$d\lambda = \sum_{\alpha=1}^{q} E_\alpha(L)\theta^\alpha \wedge dx + d\Gamma(\mathcal{C}^{(\infty)}) + \wedge^2\Gamma(\mathcal{C}^{(\infty)}), \tag{12}$$

and the forms $[\theta^\alpha \wedge dx]^{1,k}_1$ give the basis in the free module $E^{1,k}_1$.

(Here $\frac{d}{dx^I}$ are the total derivatives w.r.t. multi-index I.)

Let a Lie group G act on the manifold M. Since the G-action on $\Omega(J^\infty_k M)$ preserves the contact ideal, it induces the action on $E^{s,t}_1$.

Definition 7.1. *We shall say that $\lambda \in \Omega^k(J^\infty_k M)$ represents an invariant variational problem if $[\lambda]^{0,k}_1$ is G-invariant.*

It can be shown that for every invariant variational problem $[\lambda]^{0,k}_1$ there exists a differential form $\bar\lambda = \bar L dy^1 \wedge\cdots\wedge dy^k \in \Omega^k(\overline{J^\infty_k M})$ such that $[\lambda]^{0,k}_1 = [\mathfrak{p}^*\bar\lambda]^{0,k}_1$. The form $\bar\lambda$ in its turn defines a variational problem on $\overline{J^{r_o}_k M}$ (that is a class in $\bar E^{0,k}_1$ of the Vinogradov spectral sequence of $\overline{J^\infty_k M}$). Therefore it is desirable to understand $\mathrm{EL}(\lambda)$ in terms of the calculus of variations on the reduced jet space $\overline{J^\infty_k M}$.

It is well-known [12, 10] that in every small neighborhood of $\overline{J^\infty_k M}$ there exist functions $(y^1,..,y^k,v^1,..,v^{\bar q})$ such that any other differential invariant is a function of the y^i and the total derivatives

$$v^a_I = \frac{d^{|I|}v^a}{dy^I}$$

of v^a w.r.t. y^i (see Lemma 4.2). Moreover, as a consequence of Theorem 1, we have

$$(\overline{J^\infty_k M}, \overline{\mathcal{C}^{(\infty)}}) = (\overline{J^{r_o}_k M}, \overline{\mathcal{C}^{(r_o)}})^{(\infty)}_k = (\bar{\Delta}^{(\infty)}, \mathcal{C}^{(\infty)}),$$

where $\bar{\Delta}^{(\infty)} \overset{\iota_\infty}{\hookrightarrow} J^\infty_k \mathbb{R}^{k+\bar{q}}$ is the infinite prolongation of a certain PDE system $\bar{\Delta} \hookrightarrow J^{r_o}_k \mathbb{R}^{k+\bar{q}}$.

In local coordinates the Euler-Lagrange equations on

$$\overline{J^\infty_k M} = \bar{\Delta}^{(\infty)} \overset{\iota_\infty}{\hookrightarrow} J^\infty_k \mathbb{R}^{k+\bar{q}}$$

may be written in the same fashion as on the unconstrained jet space $J^\infty_k \mathbb{R}^{k+\bar{q}}$. More precisely, given a variational problem

$$\bar{\lambda} = \bar{L} dy^1 \wedge \cdots \wedge dy^k \in \Omega^k(\overline{J^\infty_k M}),$$

one can find a function $L_1(y^i, v^a_I) \in C^\infty(J^\infty_k \mathbb{R}^{k+\bar{q}})$ such that its restriction to $\bar{\Delta}^{(\infty)}$ is equal to $\bar{L}$ $(\iota^*_\infty L_1 = \bar{L})$, then

$$d\bar{\lambda} = \sum_{a=1}^{\bar{q}} \bar{E}_a(\bar{L}) \bar{\theta}^a \wedge dy + d\Gamma(\overline{\mathcal{C}^{(\infty)}}) + \wedge^2 \Gamma(\overline{\mathcal{C}^{(\infty)}}), \tag{13}$$

where $\bar{\theta}^a \overset{\text{def}}{=} dv^a - v^a_i dy^i$, $dy = dy^1 \wedge \cdots \wedge dy^k$, and the expression staying in the place of the Euler-Lagrange operator is defined as

$$\bar{E}_a(\bar{L}) \overset{\text{def}}{=} \iota^*_\infty \left(\sum_I (-1)^{|I|} \frac{d^{|I|}}{dy^I} \frac{\partial L_1}{\partial v^a_I} \right), \tag{14}$$

and depends on the particular choice of the function L_1.

Denote the "nonsingular" reduced infinite jet space as

$$\bar{U}^\infty \overset{\text{def}}{=} \overline{J^\infty_k M} \smallsetminus \mathfrak{p} \circ (\pi^\infty_{r_o})^{-1}(X^{r_o}).$$

Theorem 4. *Suppose that the main assumptions* **1-3***hold with regard to the trivial EDS* $(M, <0>)$. *Then there exist total differential operators on the reduced jet space* $\hat{A}^a_\alpha : C^\infty(\bar{U}^\infty) \to C^\infty(\bar{U}^\infty)$,

$$\hat{A}^a_\alpha = \sum_{0 \le |I| \le r_o - 1} A^{aI}_\alpha \frac{d^{|I|}}{dy^I},$$

(here $A^{aI}_\alpha \in C^\infty(\bar{U}^\infty)$, $\alpha = 1, .., q = \dim M - k$, $a = 1, .., \bar{q}$*) such that every invariant variational problem* $[\lambda]^{0,k}_1 = [\mathfrak{p}^* \bar{L} dy^1 \wedge \cdots \wedge dy^k]^{0,k}_1$ *has its Euler-Lagrange system as*

$$\mathrm{EL}(\lambda) = \mathfrak{p}^{-1} \left(\left\{ \sum_{a=1}^{\bar{q}} \hat{A}^a_\alpha \bar{E}_a(\bar{L}) = 0, \quad \alpha = 1, .., q \right\} \right), \tag{15}$$

where $\bar{E}_a$ *are the Euler-Lagrange operators (14) on the reduced jet space.*

Remark 7.2. Despite the fact that the Euler-Lagrange expressions defined in the formula (14) depend on the choice of the Lagrangian L_1 (if the number of independent variables is bigger than one), the expression $\sum_{a=1}^{\bar{q}} \hat{A}^a_\alpha \bar{E}_a(\bar{L})$ does not depend on this freedom.

A practical algorithm for computing the operators $\hat{A}^a_\alpha$ and some examples of computations are given in paper [5].

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